INSTITUTE FOR RISK AND RELIABILITY

Leibniz Universität Hannover

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Content

1. Introduction

2. Dimension-Reduced Probability Density Evolution Equation (DR-PDEE)

3. One-Dimensional (1D) DR-PDEE

4. Concluding Remarks

1. Introduction： Uncertainty Propagation

Input Uncertainty → **High-Dimensional Strongly-Nonlinear** → **Response Statistics**

$$\mathcal{D}\left[\boldsymbol{X}(t),\boldsymbol{\Theta}\right]=\boldsymbol{\xi}(t)$$

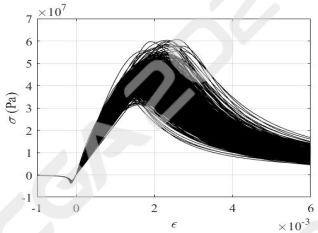
Aleatory

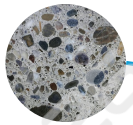
Excitation





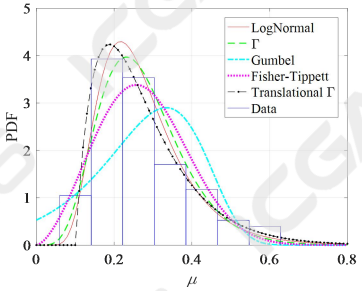
Material property

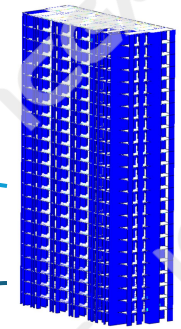


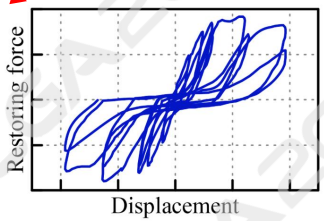


Epistemic

Incomplete control & observation







$D^{\alpha} f(t)$

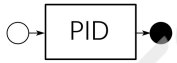
TLCD



Base isolator



Control system

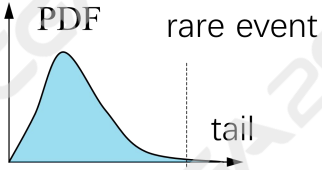


Fractional derivative

Response Statistics

Tail-accuracy

Engineering reliability







Uncertainty Propagation in Complex Stochastic Dynamic Systems

Data-Driven/Statistical

unbiased

Monte Carlo Simulation (MCS)

- **Low-discrepancy sequences** (Niederreiter 1992)
- **Variance reduction techniques**
  - Importance sampling (Harbitz 1983)
  - Subset simulation (Au & Beck 2001)
  - Line sampling (Schuëller et al., 2004)
  - …

biased

Moment Based Methods (He & Gong 2016, Xu et al. 2016)

Surrogate Models (Jensen et al. 2017, Wang et al. 2019)

Physics-driven

Random Vibration

- Fokker–Planck–Kolmogorov (FPK) equation — exact
- Stochastic averaging (Stratonovich 1963) — approximate
- Statisitcal linearization (Roberts & Spanos 2003) — approximate
- Wiener path integral (Kougioumtzoglou et al. 2024) — approximate

Stochastic Finite Element

- Stochastic perturbation (Soong & Bogdanoff 1964) — approximate
- Polynomial expansion (Spanos & Ghanem 1989, Li 1995) — approximate



Physics-Driven Data-Assisted Approaches

2. Dimension-Reduced Probability Density Evolution Equation (DR-PDEE)

Theoretical Derivation

Total probability theorem

$$p(y,t)=\int_{-\infty}^{+\infty} p(y,t|y',t')\,p(y',t')\,\mathrm{d}y'$$

$Y(t)$ 1-Dimensional Stochastic Process

Kramers-Moyal expansion

(Kramers 1940, Moyal 1949)

$$\frac{\partial p(y,t)}{\partial t}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial y^n}\left[a_n(y,t)\,p(y,t)\right]$$

***n*-th conditional derivative moments**

$$\alpha_n(y,t)=\lim_{\Delta t\to 0}\mathrm{E}\left[\Delta Y^n(t)/\Delta t\,|\,Y(t)=y\right]$$

Path continuous condition

$$\lim_{\tau\to 0}\frac{1}{\tau}\int_{|y-y'|>\varepsilon} p(y,t'+\tau|y',t')\,\mathrm{d}y=0$$

$$\begin{aligned}|\alpha_n(y,t)| &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}|\xi|^n\,p(y+\xi,t+\Delta t|y,t)\,\mathrm{d}\xi\\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\varepsilon}^{\varepsilon}|\xi|^n\,p(y+\xi,t+\Delta t|y,t)\,\mathrm{d}\xi\\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\varepsilon}^{\varepsilon}|\xi|^{n-2}\,\xi^2\,p(y+\xi,t+\Delta t|y,t)\,\mathrm{d}\xi\\ &\le \varepsilon^{n-2}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\varepsilon}^{\varepsilon}\xi^2\,p(y+\xi,t+\Delta t|y,t)\,\mathrm{d}\xi.\end{aligned}$$

$$\alpha_n(y,t)=0,\quad \text{for}\quad n\ge 3$$

Lyu M.Z. & Chen J.B. 2022. *Structural Safety* 98: 102233.

2. Dimension-Reduced Probability Density Evolution Equation (DR-PDEE)

Theoretical Derivation

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$$\alpha_n(y,t)=0,\quad \text{for}\quad n\geq 3$$

$$\alpha_1(y,t)\to a^{(\mathrm{int})}(y,t)\quad \alpha_2(y,t)\to b^{(\mathrm{int})}(y,t)$$

DR-PDEE

(Lyu & Chen 2022)

$$\frac{\partial p(y,t)}{\partial t}=-\frac{\partial\left[a^{(\mathrm{int})}(y,t)\,p(y,t)\right]}{\partial y}+\frac{1}{2}\frac{\partial^2\left[b^{(\mathrm{int})}(y,t)\,p(y,t)\right]}{\partial y^2}$$

Exact

intrinsic drift coefficient

intrinsic diffusion coefficient

Lyu M.Z. & Chen J.B. 2022. *Structural Safety* 98: 102233.

2. Dimension-Reduced Probability Density Evolution Equation (DR-PDEE)

DR-PDEE

$$\frac{\partial p(y,t)}{\partial t} = -\frac{\partial\left[a^{(\mathrm{int})}(y,t)\,p(y,t)\right]}{\partial y} + \frac{1}{2}\frac{\partial^2\left[b^{(\mathrm{int})}(y,t)\,p(y,t)\right]}{\partial y^2}$$

1D

Governing equation of $p(y,t)$ for **any** **path-continuous** stochastic process

***m*-D** path continuous system

$D\left[\boldsymbol{X}(t),\boldsymbol{\Theta}\right] = \xi(t)$ → Stochastic excitation

→ Random parameters

Non-Markovian (e.g. **Fractional Derivative**)

$$\left[Y(t_0), Y(t_1), \cdots, Y(t_j)\right] \to Y(t_{j+1})$$

1D response of interest

$$Y(t) = g\left[\boldsymbol{X}(t)\right]$$

1D system

$$\dot{\tilde{y}} = a^{(\mathrm{int})}(\tilde{y},t) + \sqrt{b^{(\mathrm{int})}(\tilde{y},t)}\,\xi(t)$$

Exact low dimensional surrogate in probability sense

- Establish the **physical law** of uncertainty propagation in drastically reduced state space
- Provide **efficient tool** for stochastic response analysis of complex systems

DR-PDEE
(Lyu & Chen 2022)

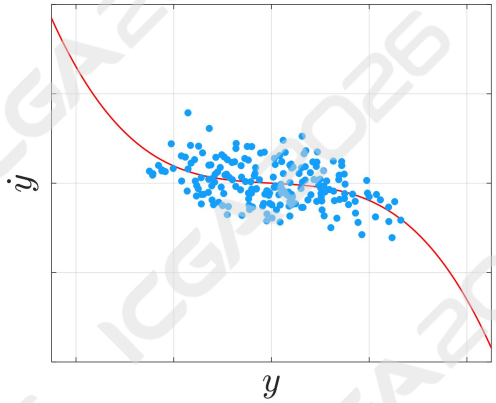
$$\frac{\partial p(y,t)}{\partial t} = -\frac{\partial\left[a^{(\mathrm{int})}(y,t)\,p(y,t)\right]}{\partial y} + \frac{1}{2}\frac{\partial^2\left[b^{(\mathrm{int})}(y,t)\,p(y,t)\right]}{\partial y^2}$$

Intrinsic drift coefficient

$$a^{(\mathrm{int})}(y,t) = \mathrm{E}\left[\dot{Y}(t)\middle|Y(t)=y\right]$$

Analytical solution only available for a few systems (Luo et al. 2023, 2025, Sun & Chen 2022)

Estimated from sample data



Physics-Driven Data-Assisted

Intrinsic diffusion coefficient

$$b^{(\mathrm{int})}(y,t) = \lim_{\Delta t\to 0}\mathrm{E}\left[\Delta Y^2(t)/\Delta t\middle|Y(t)=y\right]$$

Analytical in most cases

- No direct white noise excitation $b^{(\mathrm{int})}(y,t)=0$
- Gaussian white noise $b^{(\mathrm{int})}(y,t)=D$

Estimated from sample data

- High-frequency oscillation
- Unknown governing equation

Sun T, Chen J, 2022. *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering*, 8(2), p.04022012.

Luo Y, Lyu MZ, Chen JB, Spanos PD, 2023. *Theoretical and Applied Mechanics Letters*, 13(3), p.100436.

Luo Y, Dang C, Broggi M, Beer M, 2025. *Probabilistic Engineering Mechanics*, 79, p.103735.

Data-Assisted DR-PDEE

Numerical Implementation

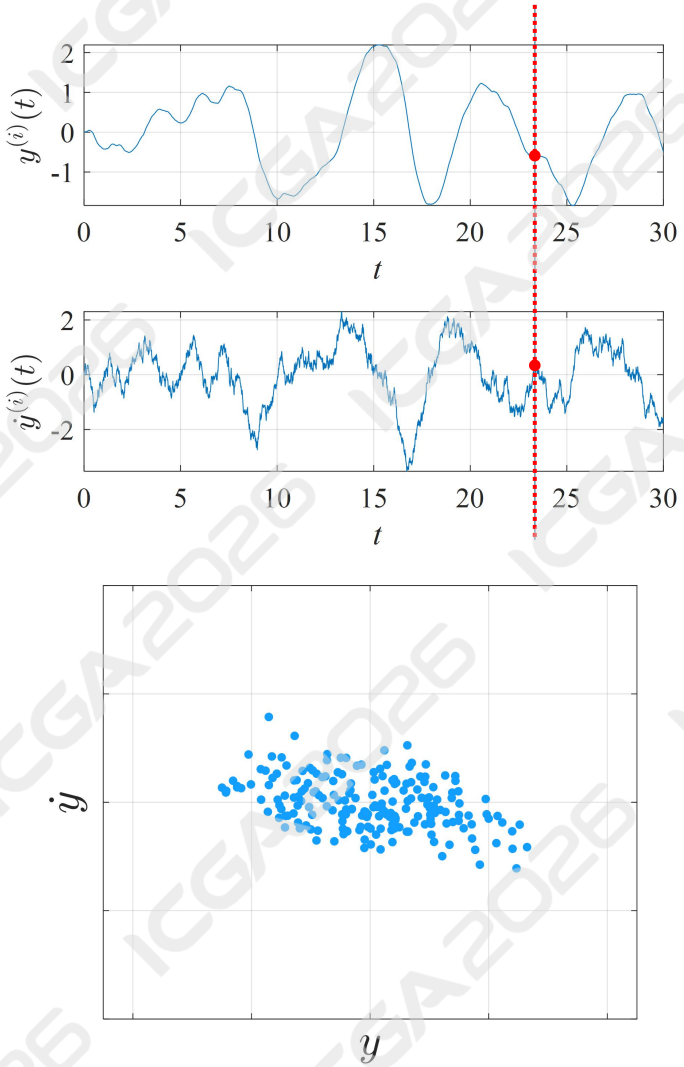
STEP 1

Deterministic analysis

$$D\left[\boldsymbol{X}(t),\boldsymbol{\Theta}\right]=\boldsymbol{\xi}(t) \quad Y(t)=g\left[\boldsymbol{X}(t)\right]$$

Obtain 100~800 samples

$$y^{(i)}(t),\dot{y}^{(i)}(t)$$



Data-Assisted DR-PDEE

Numerical Implementation

STEP 1 **Deterministic analysis**

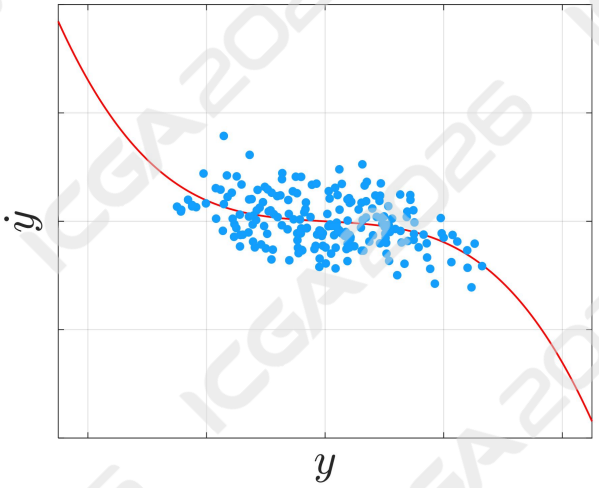
Obtain 100~800 samples

$$y^{(i)}(t), \dot{y}^{(i)}(t)$$

STEP 2 **Estimate** $a^{(\mathrm{int})}(y,t)$

Regression at each time step

- Locally weighted smoothing scatterplots (LoWeSS)



- In MATLAB

```
f = fit([x y],z,'lowess')
```

lowess_1D.m

lowess_2D.m

Data-Assisted DR-PDEE

Numerical Implementation

STEP 1 **Deterministic analysis**

Obtain 100~800 samples

$$y^{(i)}(t), \dot{y}^{(i)}(t)$$

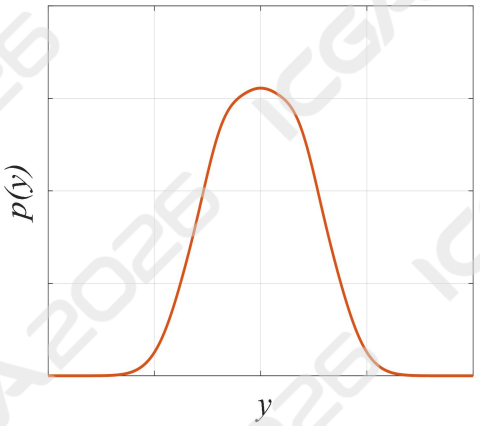
STEP 2 **Estimate** $a^{(\mathrm{int})}(y,t)$

Regression at each time step

- Locally weighted smoothing scatterplots (LoWeSS)

STEP 3 **Solve the DR-PDEE**

$$\frac{\partial p}{\partial t} = -\frac{\partial\left[a^{(\mathrm{int})}(y,t)\,p\right]}{\partial y} + \frac{1}{2}\frac{\partial^2\left[b^{(\mathrm{int})}(y,t)\,p\right]}{\partial y^2}$$



- **Classically: Path integral solution (PIS)**

  Chapman-Kolmogorov equation

$$p(y,t|y_0,t_0) = \int_{-\infty}^{+\infty} p(y,t|y',t')\,p(y',t'|y_0,t_0)\,\mathrm{d}y' \xrightarrow{\mathrm{d}t = t - t' \to 0}$$

Transition PDF

$$p(y,t|y',t') = \frac{1}{\sqrt{2\pi b^{(\mathrm{int})}(y',t')\,\mathrm{d}t}} \exp\left[-\frac{\left(y - y' - a^{(\mathrm{int})}(y',t')\,\mathrm{d}t\right)^2}{2b^{(\mathrm{int})}(y',t')\,\mathrm{d}t}\right]$$

2. DR-PDEE: Numerical example

12-DOF Linear Fractional Differential system (Luo et al. 2023)

$$\dot{\boldsymbol{Y}}(t) = -[\mathbf{C}\mathrm{D}^{\alpha}\boldsymbol{Y}(t) + \mathbf{K}\boldsymbol{Y}(t)] + \mathbf{L}\boldsymbol{\xi}(t)$$ ← Gaussian excitation

DR-PDEE for $Y_l(t)$

$$\frac{\partial p_{Y_l}(y,t)}{\partial t} = -\frac{\partial [a^{(\mathrm{int})}(y,t)\, p_{Y_l}(y,t)]}{\partial y} + \frac{1}{2}\frac{\partial^2 [b^{(\mathrm{int})}(y,t)\, p_{Y_l}(y,t)]}{\partial y^2}$$

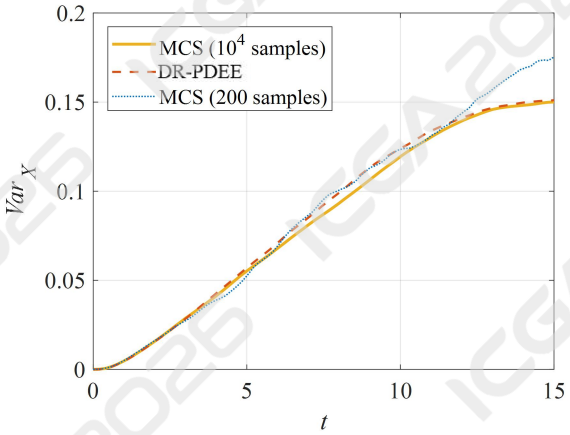
Intrinsic drift coefficient

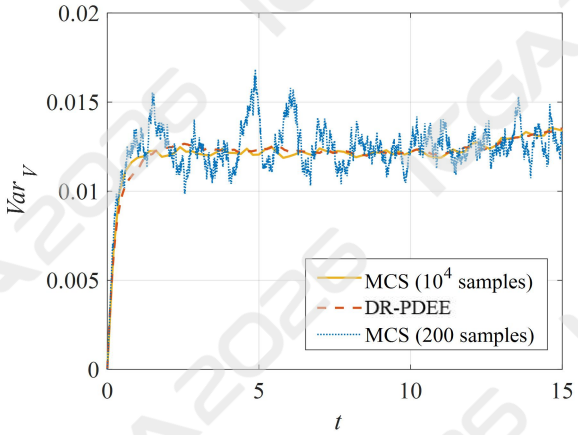
$$a^{(\mathrm{int})}(y,t) = -\frac{y-\mu_l(t)}{\sigma_l^2(t)}[\boldsymbol{C}_{(l,\cdot)}\mathrm{D}_1^{\alpha}\tilde{\boldsymbol{\varsigma}}_{(\cdot,l)}(t,t) + \boldsymbol{K}_{(l,\cdot)}\boldsymbol{\varsigma}_{(\cdot,l)}(t)] - \boldsymbol{C}_{(l,\cdot)}\mathrm{D}^{\alpha}\boldsymbol{\mu}(t) - \boldsymbol{K}_{(l,\cdot)}\boldsymbol{\mu}(t)$$

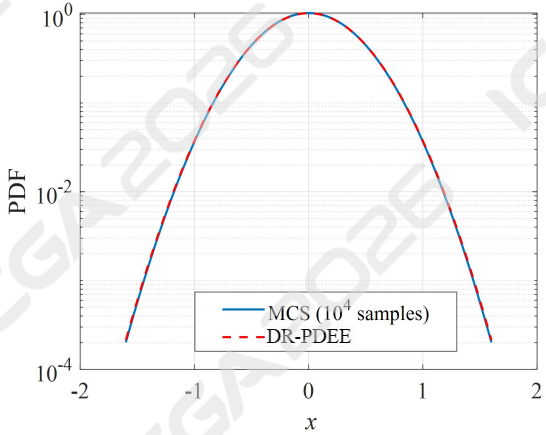
← Linear function

Intrinsic diffusion

$$b^{(\mathrm{int})}(y,t) = \boldsymbol{L}_{(l,\cdot)}\boldsymbol{L}_{(l,\cdot)}^{\mathrm{T}} = b_{ll},$$



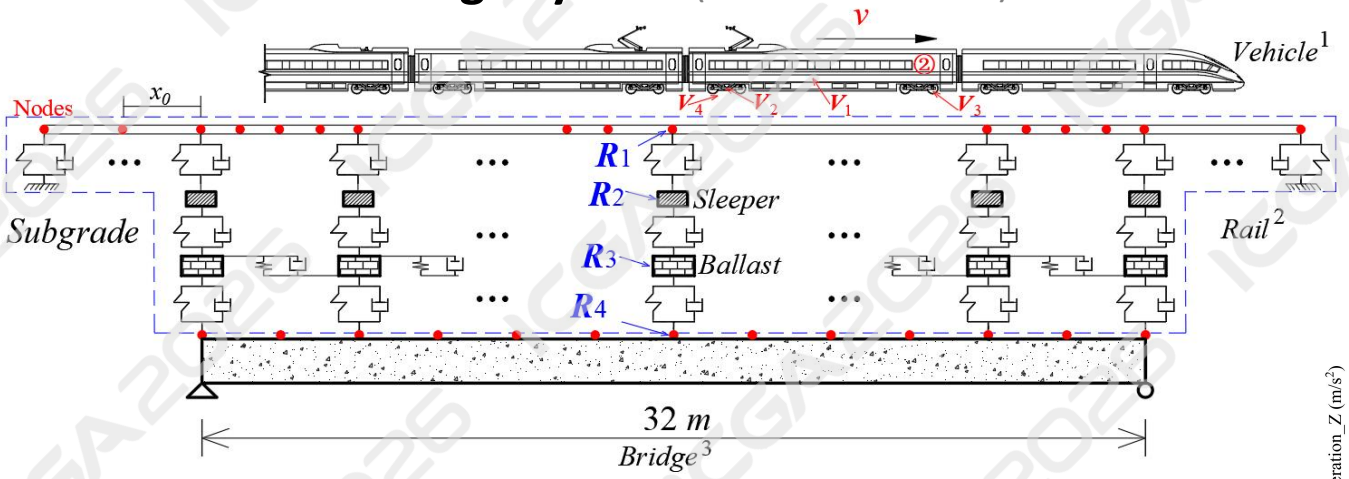




Luo, Y., Lyu, M.Z., Chen, J.B. and Spanos, P.D., 2023. *Theoretical and Applied Mechanics Letters*, *13*(3), p.100436.

2. Dimension-Reduced Probability Density Evolution Equation (DR-PDEE)

3D Vehicle-Rail-Bridge System (Yuan & Luo 2026)

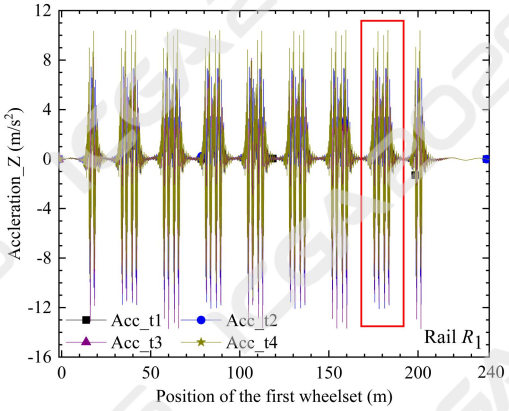


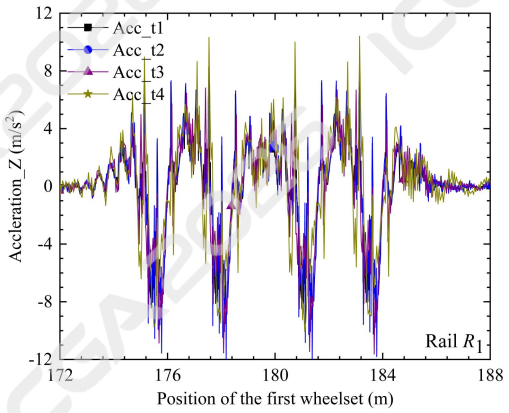
DOFs

$51\times8+$

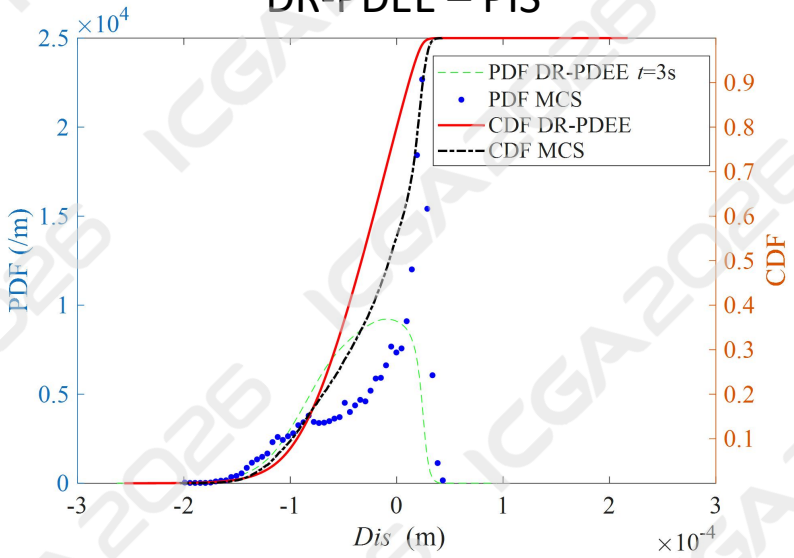
$339\times20+$

$65\times6-5=7573$





DR-PDEE – PIS



$t_1 = 0.0001$ s

$t_2 = 0.0002$ s

$t_3 = 0.0004$ s

$t_4 = 0.0005$ s

T = 3.0138 s

68 minutes for 1 simulation

Strongly non-stationary non-Gaussian

2. Dimension-Reduced Probability Density Evolution Equation (DR-PDEE)

Numerical Implementation

STEP 1 **Deterministic analysis**

$$D\left[\boldsymbol{X}(t),\boldsymbol{\Theta}\right]=\xi(t) \quad Y(t)=g\left[\boldsymbol{X}(t)\right]$$

Obtain 100~800 samples

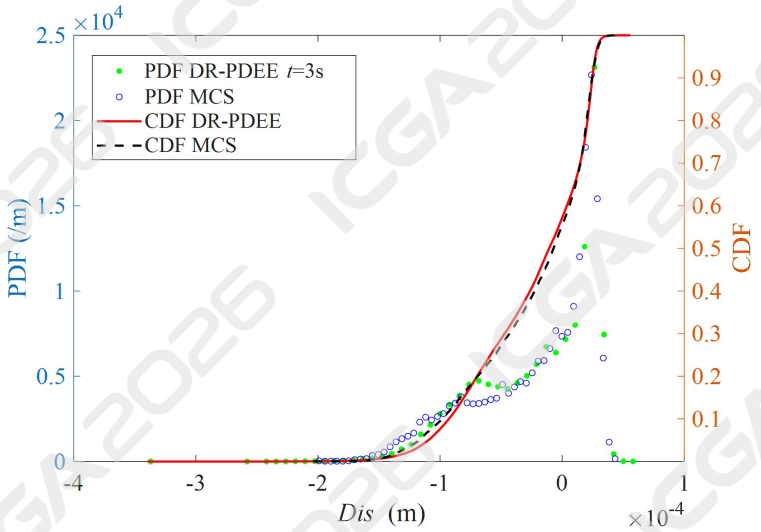
$$y^{(i)}(t),\dot{y}^{(i)}(t)$$

STEP 2 **Estimate** $a^{(\mathrm{int})}(y,t)$

Regression at each time step

- Locally weighted smoothing scatterplots (LoWeSS)

DR-PDEE – DR-MCS



STEP 3 **Solve the DR-PDEE**

$$\frac{\partial p}{\partial t}=-\frac{\partial\left[a^{(\mathrm{int})}(y,t)\,p\right]}{\partial y}+\frac{1}{2}\frac{\partial^2\left[b^{(\mathrm{int})}(y,t)\,p\right]}{\partial y^2}$$

Classical: Path integration

Proposed:

$$\dot{y}=a^{(\mathrm{int})}(y,t)+\sqrt{b^{(\mathrm{int})}(y,t)}\,\xi(t)$$

Dimension reduced Monte Carlo (DR-MC)

(Yuan & Luo 2026)

3. One-Dimensional (1D) DR-PDEE

DR-PDEE

$$\frac{\partial p(y,t)}{\partial t} = -\frac{\partial\left[a^{(\mathrm{int})}(y,t)\,p(y,t)\right]}{\partial y} + \frac{1}{2}\frac{\partial^2\left[b^{(\mathrm{int})}(y,t)\,p(y,t)\right]}{\partial y^2}$$

$b^{(\mathrm{int})}(y,t) \neq 0$ is **required** for Path integration

- no *direct* white noise excitation $b^{(\mathrm{int})}(y,t)=0$

1D DR-PDEE with no diffusion

$$\frac{\partial p(y,t)}{\partial t} = -\frac{\partial\left[a^{(\mathrm{int})}(y,t)\,p(y,t)\right]}{\partial y}$$

- **Path integration** ✕

$$p\left(y,t\middle|y',t'\right) = \delta\left(y - y' - a^{(\mathrm{int})}(y',t')\,\mathrm{d}t\right)$$

Introduce numerical diffusion → **large error**

- **Dimension-reduce Monte Carlo (DR-MC)** ?

$$\dot{y} = a^{(\mathrm{int})}(y,t)$$

Classical Scheme

Introduce Gaussian white noise term

- Filter approximation (Luo et al. 2022)
- Auxiliary diffusion process (Sun et al. 2025)

$$\begin{cases} \dot{y}_1 = a_1^{(\mathrm{int})}(y_1,y_2,t) & \rightarrow \text{Original dimension} \\ \dot{y}_2 = a_2^{(\mathrm{int})}(y_1,y_2,t) + \sqrt{D}\,\xi(t) & \rightarrow \text{Filter/Auxiliary dimension} \end{cases}$$

2D DR-PDEE

$$\frac{\partial p(y_1,y_2,t)}{\partial t} = -\frac{\partial\left[a_1^{(\mathrm{int})}(y_1,y_2,t)\,p(y_1,y_2,t)\right]}{\partial y_1} - \frac{\partial\left[a_2^{(\mathrm{int})}(y_1,y_2,t)\,p(y_1,y_2,t)\right]}{\partial y_2} + \frac{1}{2}\frac{\partial^2\left[Dp(y_1,y_2,t)\right]}{\partial y_2^2}$$

- **Estimate 2D** $a_1^{(\mathrm{int})}(y_1,y_2,t)$ **&** $a_2^{(\mathrm{int})}(y_1,y_2,t)$
- **Solve 2D PDE**

Luo Y, Chen JB, Spanos PD, 2022. *Probabilistic Engineering Mechanics*, 67:103197.
Sun TT, Chen JB, **Luo Y**, Lyu JH, 2025. *Computer Methods in Applied Mechanics and Engineering*, 442:118029.

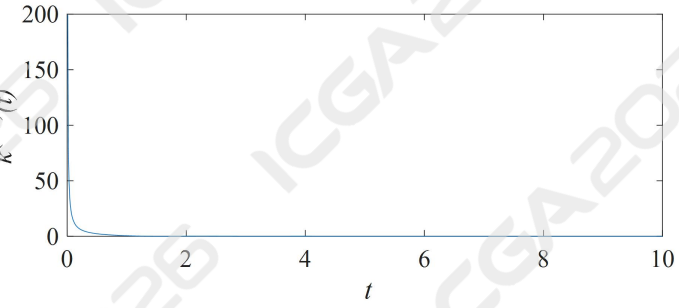
3. One-Dimensional (1D) DR-PDEE: Numerical Validation

➢ 1:

$$\ddot{X}(t)+2\zeta\omega\dot{X}(t)+\omega^2 X(t)=\xi(t)$$

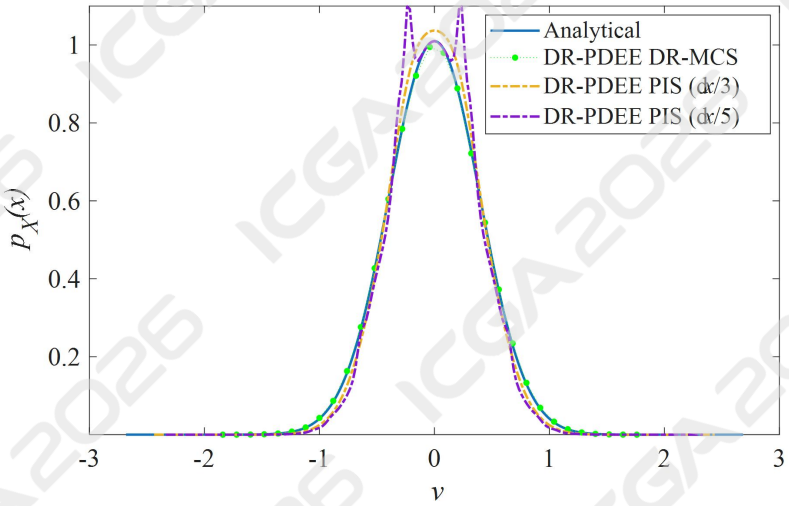
$$\zeta=0.2,\ \omega=2,\ D=1 \qquad \mathrm{E}\left[\xi(t)\xi(t+\tau)\right]=D\delta(t)$$

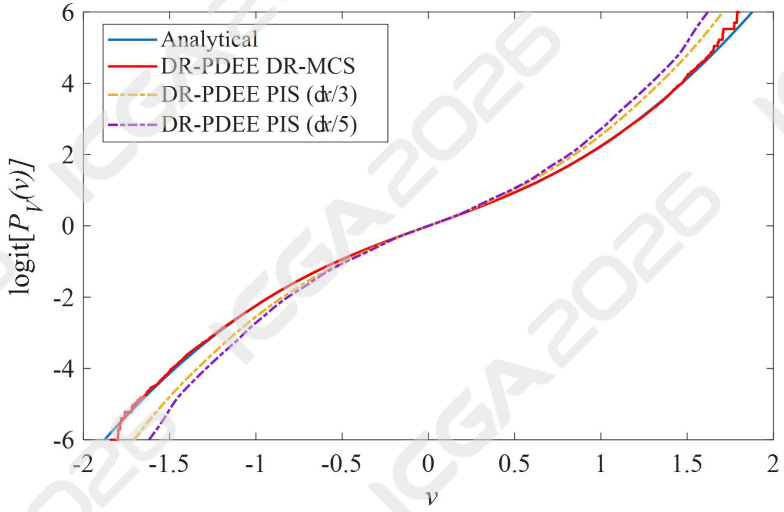
- **Purpose:** Study the accuracy of **DR-MCS** for **1D DR-PDEE** with $b^{(\mathrm{int})}(y,t)=0$
- **We can derive analytical** $a_x^{(\mathrm{int})}(x,t)=k^{(\mathrm{int})}(t)x$



- **Conclusion**

  DR-MCS gives **robust and accurate solution,** while **PIS cannot**





3. One-Dimensional (1D) DR-PDEE: Numerical Validation

➢ **2**:

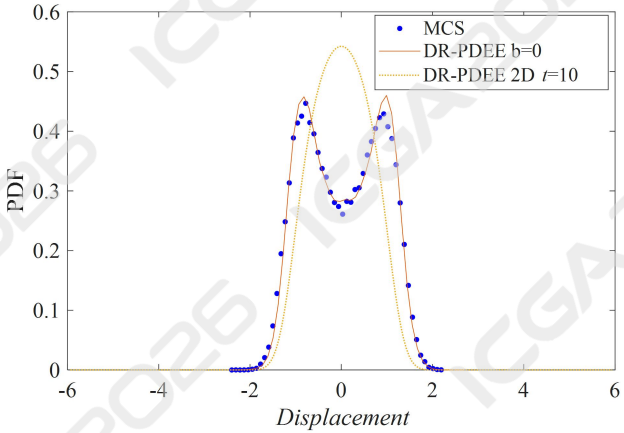
$$\ddot{X}(t)+2\zeta_0\omega_0\dot{X}(t)+\omega_0^2\left[k_1X(t)+k_2X^3(t)\right]=f(t) \qquad S_f(\omega)=\frac{D}{\omega^2+\omega_s^2}$$

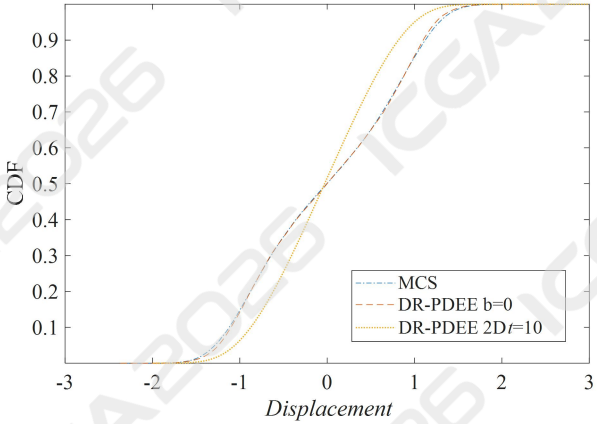
$$\zeta_0=0.5,\ \omega_0=1,\ k_1=-1,\ k_2\sim\mathrm{U}(1,2),\ D=1,\ \omega_s=2$$

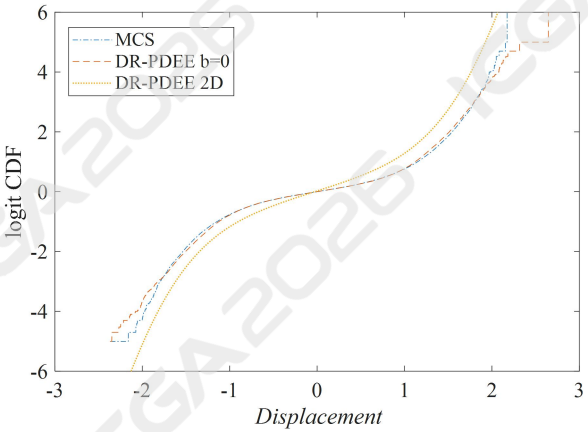
- **Case 1: Displacement** $X(t)$

Analytical $b^{(\mathrm{int})}(y,t)=0$

- **1D DR-PDEE** DR-MCS √
- **2D DR-PDEE** PIS ×







3. One-Dimensional (1D) DR-PDEE: Numerical Validation

➢ **2**:

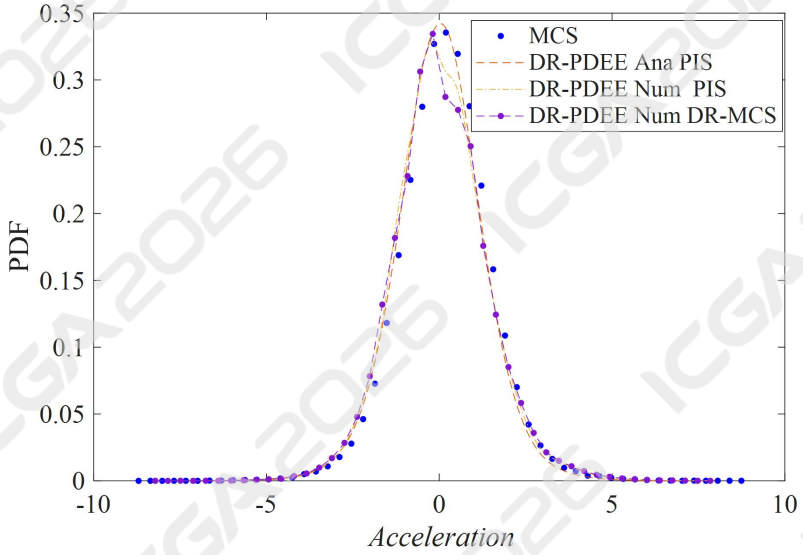
$$\ddot{X}(t)+2\zeta_0\omega_0\dot{X}(t)+\omega_0^2\left[k_1X(t)+k_2X^3(t)\right]=f(t) \qquad S_f(\omega)=\frac{D}{\omega^2+\omega_s^2}$$

$$\zeta_0=0.5,\ \omega_0=1,\ k_1=-1,\ k_2\sim\mathrm{U}(1,2),\ D=1,\ \omega_s=2$$

- **Case 2: Acceleration** $\ddot{X}(t)$

Analytical $b^{(\mathrm{int})}(y,t)=D$ ▪ **1D DR-PDEE** PIS √

Estimated $b^{(\mathrm{int})}(y,t)$ ▪ **1D DR-PDEE** PIS ▪ **1D DR-PDEE** DR-MCS



3. One-Dimensional (1D) DR-PDEE: Numerical Validation

➢ 3: 10-

$$\mathbf{M}\ddot{\boldsymbol{X}}(t)+\mathbf{C}\dot{\boldsymbol{X}}(t)+\boldsymbol{f}\left[\boldsymbol{X}(t),\boldsymbol{\Theta}\right]=\boldsymbol{B}\xi(t)$$

$$\mathbf{M}=\mathrm{diag}\left[3,3,3,3,2,2,2,2,1.5,1.5\right]\times10^{5},$$

$$\mathbf{C}=a_0\mathbf{M}+a_1\mathbf{K}(\boldsymbol{\Theta}),\quad\begin{pmatrix}a_0\\a_1\end{pmatrix}=\frac{2\zeta}{\omega_1+\omega_3}\begin{Bmatrix}\omega_1\omega_3\\1\end{Bmatrix},\quad\zeta=0.05,$$

$$\mathbf{K}(\boldsymbol{\Theta})=\begin{bmatrix}\theta_1k_1+\theta_2k_2 & -\theta_2k_2 & & & \\ -\theta_2k_2 & \theta_2k_2+\theta_3k_3 & -\theta_3k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -\theta_9k_9 & \theta_9k_9+\theta_{10}k_{10} & -\theta_{10}k_{10} \\ & & & -\theta_{10}k_{10} & \theta_{10}k_{10}\end{bmatrix},$$

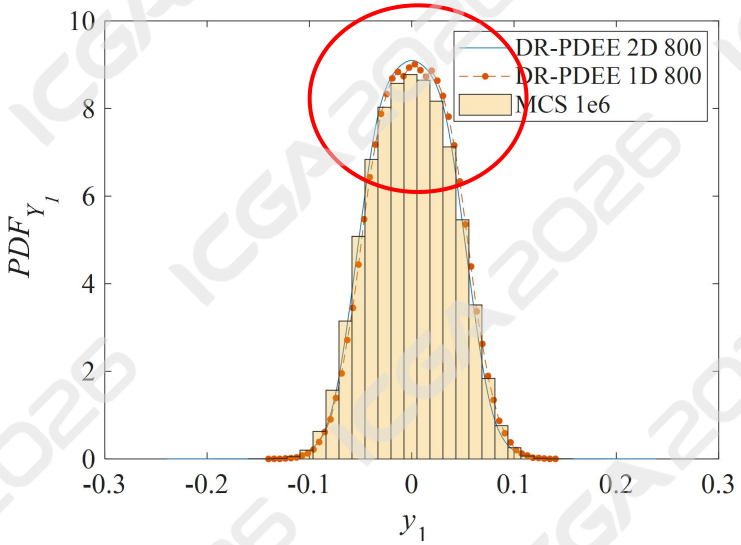
$$\left[k_1,\cdots,k_{10}\right]=\left[5,5,5,5,4,4,4,4,3,3\right]\times10^{7},\quad\left[\theta_1,\theta_2,\cdots,\theta_{10}\right]\sim LN$$

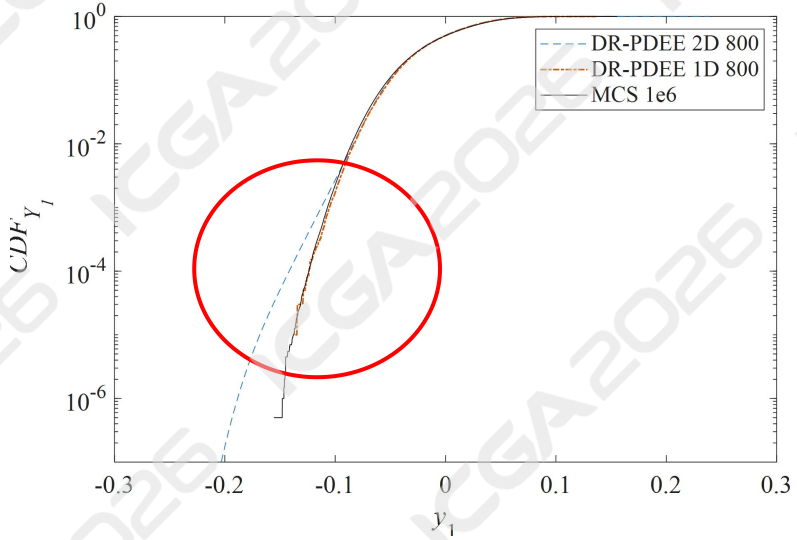
$$\boldsymbol{B}=\left[1,\cdots,1\right]^{\mathrm{T}},\quad R_{\xi}(\tau)=D_0\delta(\tau),\ D_0=0.2.$$

$$\boldsymbol{f}\left[\boldsymbol{X}(t),\boldsymbol{\Theta}\right]=\left[f_1^*-f_2^*,f_2^*-f_3^*,\cdots,f_{n-1}^*-f_n^*,f_n^*\right],$$

$$f_j^*=k_j(\theta_j)\left[\hat{X}_j+\frac{\varepsilon}{\sigma_s^2}\hat{X}_j^3\right],\quad\varepsilon=0.72,$$

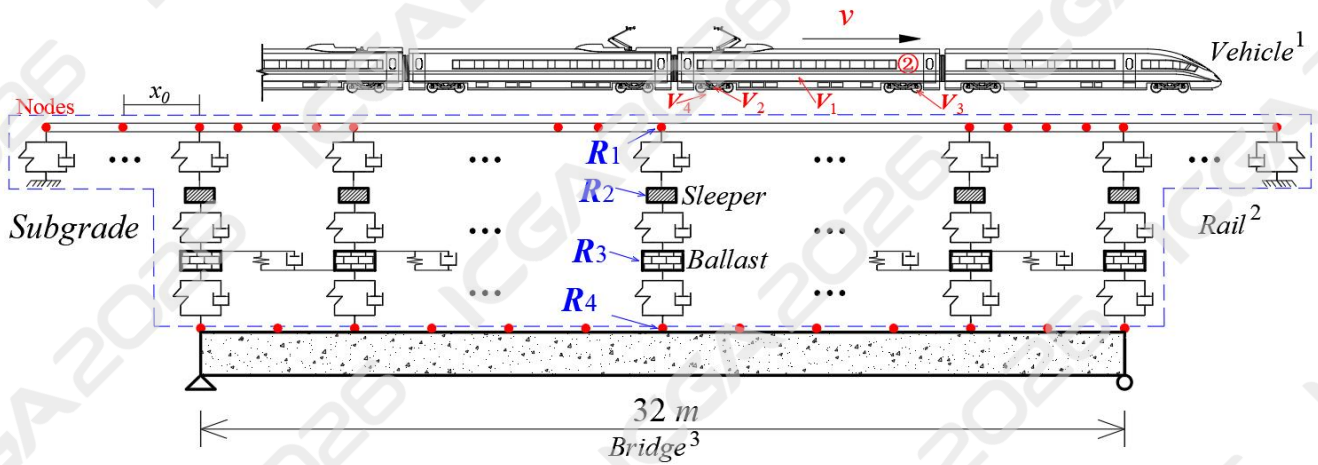
$$\hat{\boldsymbol{X}}=\left[X_1,X_2-X_1,\cdots,X_n-X_{n-1}\right]^{\mathrm{T}},$$





3. One-Dimensional (1D) DR-PDEE: Numerical Validation

➤ 4: - -



DOFs

$51\times 8+$

$339\times 20+$

$65\times 6-5=7573$

Key response of Interest: Displacement & Acceleration at Bridge Midspan

$t_1 = 0.0001$ s Acceleration: 68 minutes for 1 simulation

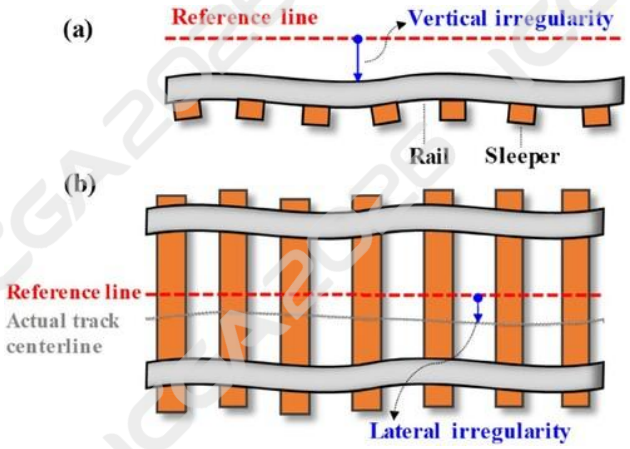
$t_4 = 0.0005$ s Displacement: 15 minutes for 1 simulation

T = 3.0138 s

Random Sources

- *25 Random Structural Parameters*
- *Track irregularities in 2 directions*

German track spectra (Zhai 2019)

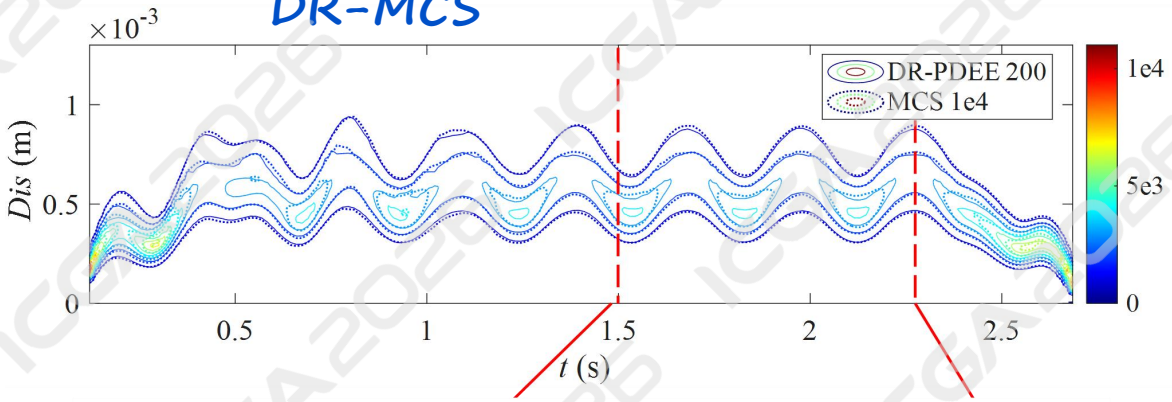


3. One-Dimensional (1D) DR-PDEE: Numerical Validation

➢ 4: - -

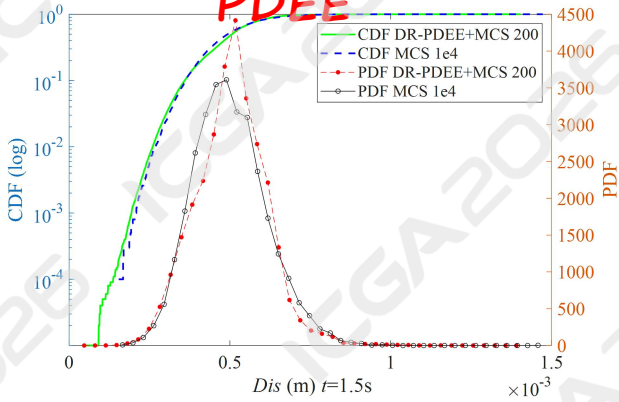
- Displacement (Validated against 1e4 MCS)

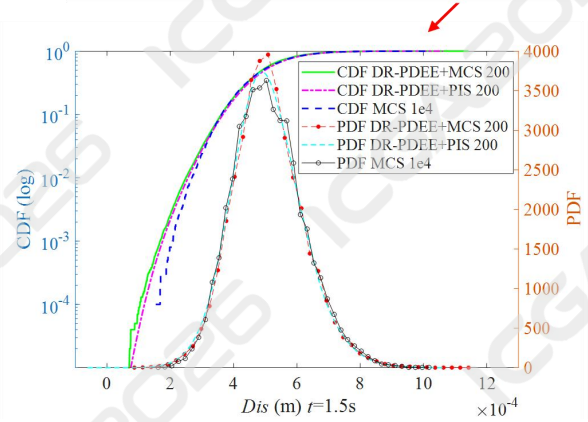
2D DR-PDEE PIS vs. DR-MCS



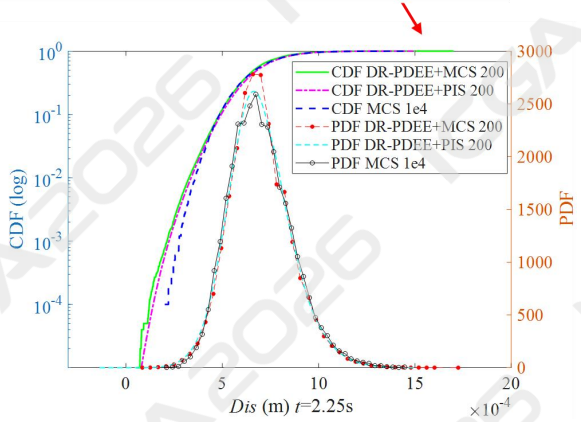
(c) Comparison of the time-varying PDF contours: DR-PDEE vs. MCS

1D DR-PDEE

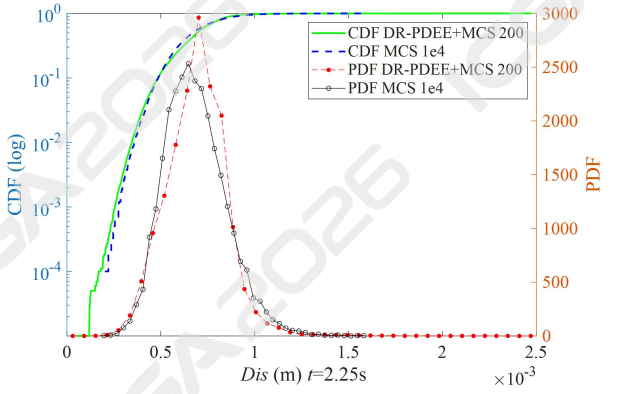




(d) PDFs and CDFs at t = 1.5 s



(e) PDFs and CDFs at t = 2.25 s

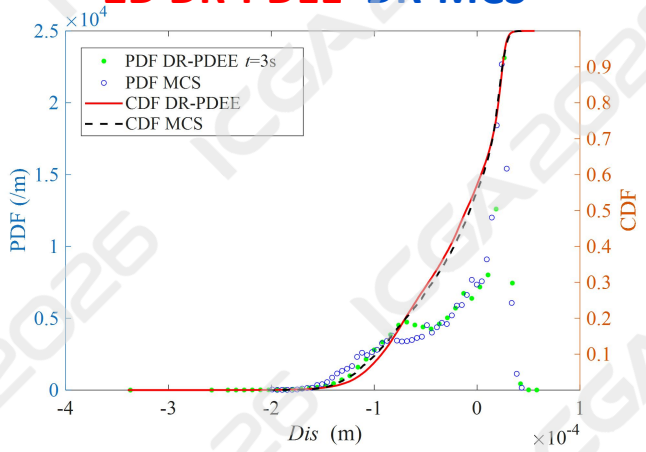


3. One-Dimensional (1D) DR-PDEE: Numerical Validation

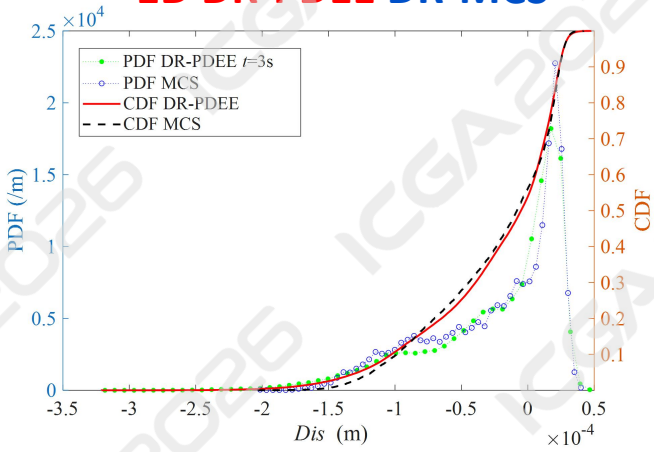
➢ 4: - -

- *Displacement (Validated against 1e4 MCS)*

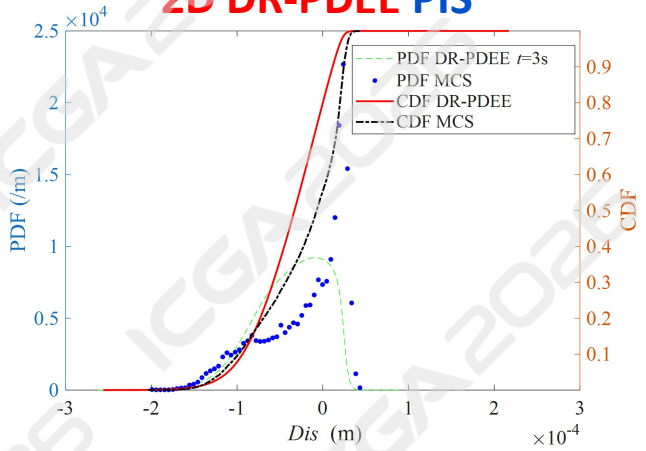
2D DR-PDEE DR-MCS



1D DR-PDEE DR-MCS



2D DR-PDEE PIS



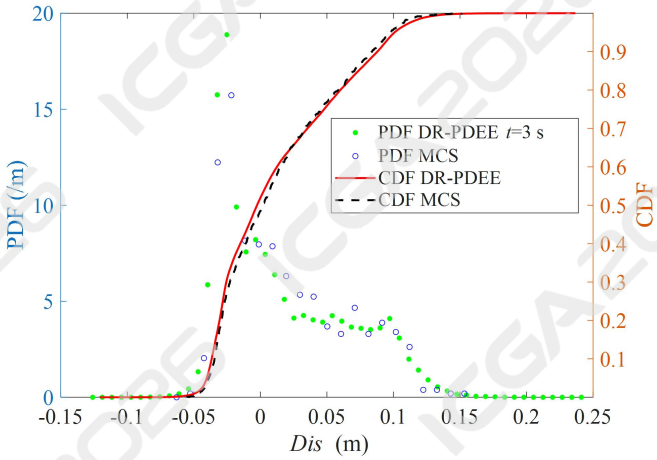
3. One-Dimensional (1D) DR-PDEE: Numerical Validation

➢ 4: - -

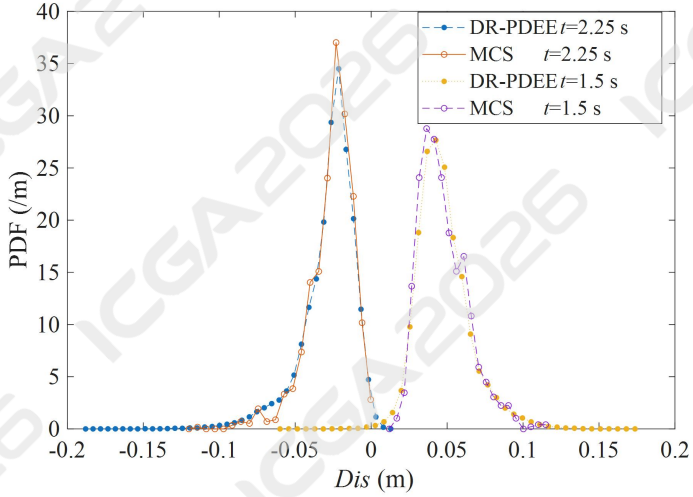
- Acceleration (Validated against 1e3 MCS)

$b^{(\mathrm{int})}(y,t)$

MCS 1000



DR-PDEE 1000



4. Concluding Remarks

DR-PDEE

$$\frac{\partial p(y,t)}{\partial t} = -\frac{\partial\left[a^{(\mathrm{int})}(y,t)\,p(y,t)\right]}{\partial y} + \frac{1}{2}\frac{\partial^2\left[b^{(\mathrm{int})}(y,t)\,p(y,t)\right]}{\partial y^2}$$

1D

Governing equation of $p(y,t)$ for **any** **path-continuous** stochastic process

***m*-D** path continuous system

$$D\left[\boldsymbol{X}(t),\boldsymbol{\Theta}\right] = \boldsymbol{\xi}(t)$$

→ **1D** response of interest

$$Y(t) = g\left[\boldsymbol{X}(t)\right]$$

1D system

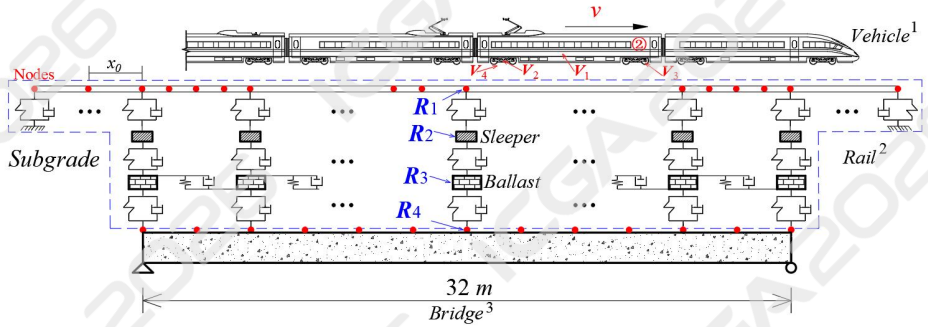
$$\dot{\tilde{y}} = a^{(\mathrm{int})}(\tilde{y},t) + \sqrt{b^{(\mathrm{int})}(\tilde{y},t)}\,\xi(t)$$

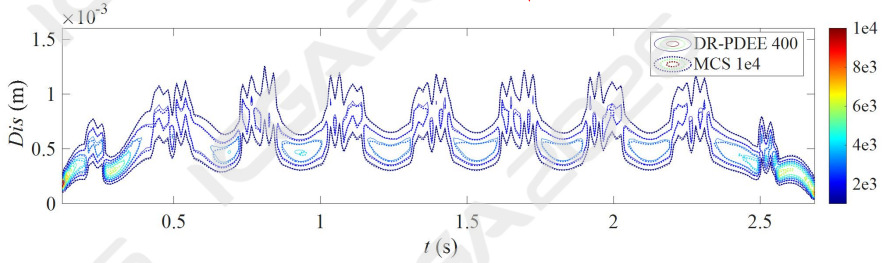
Dimension-Reduced Monte Carlo

$b^{(\mathrm{int})}(y,t) \neq 0$ & $b^{(\mathrm{int})}(y,t) = 0$

Exact low dimensional surrogate in probability sense

3D Vehicle-Rail-Bridge System





INSTITUTE FOR RISK AND RELIABILITY

Leibniz Universität Hannover

Thank you for your attention!

