Isotropic shallow shells of maximum stiffness

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Outline

- Introduction
- Optimum design problem
  - original
  - relaxed
- Example

Introduction



St. Aloysius Church in Jackson, NJ, designed by Erdy McHenry Architecture (2009)
Source: em-arc.com

Real-world motivation

Thin shell structures
carrying combined membrane and bending loads

Introduction



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Ω design domain

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Ω	design domain
$\chi \colon \Omega \to \{0, 1\}$	material distribution in Ω

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Introduction



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Ω	design domain
$\chi \colon \Omega \to \{0, 1\}$	material distribution in Ω
$\sigma, \eta \colon \Omega \to \mathcal{M}^{2\times 2}_{sym}$	membrane forces and bending moments in equilibrium with the load

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Optimum design problem

(original formulation)

Optimum design problem – original formulation

The objective

For the given load, distribute the material
to make the structure as stiff as possible at a given material ‘cost’

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Optimum design problem – original formulation

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to make the structure as stiff as possible at a given material ‘cost’

Mathematically

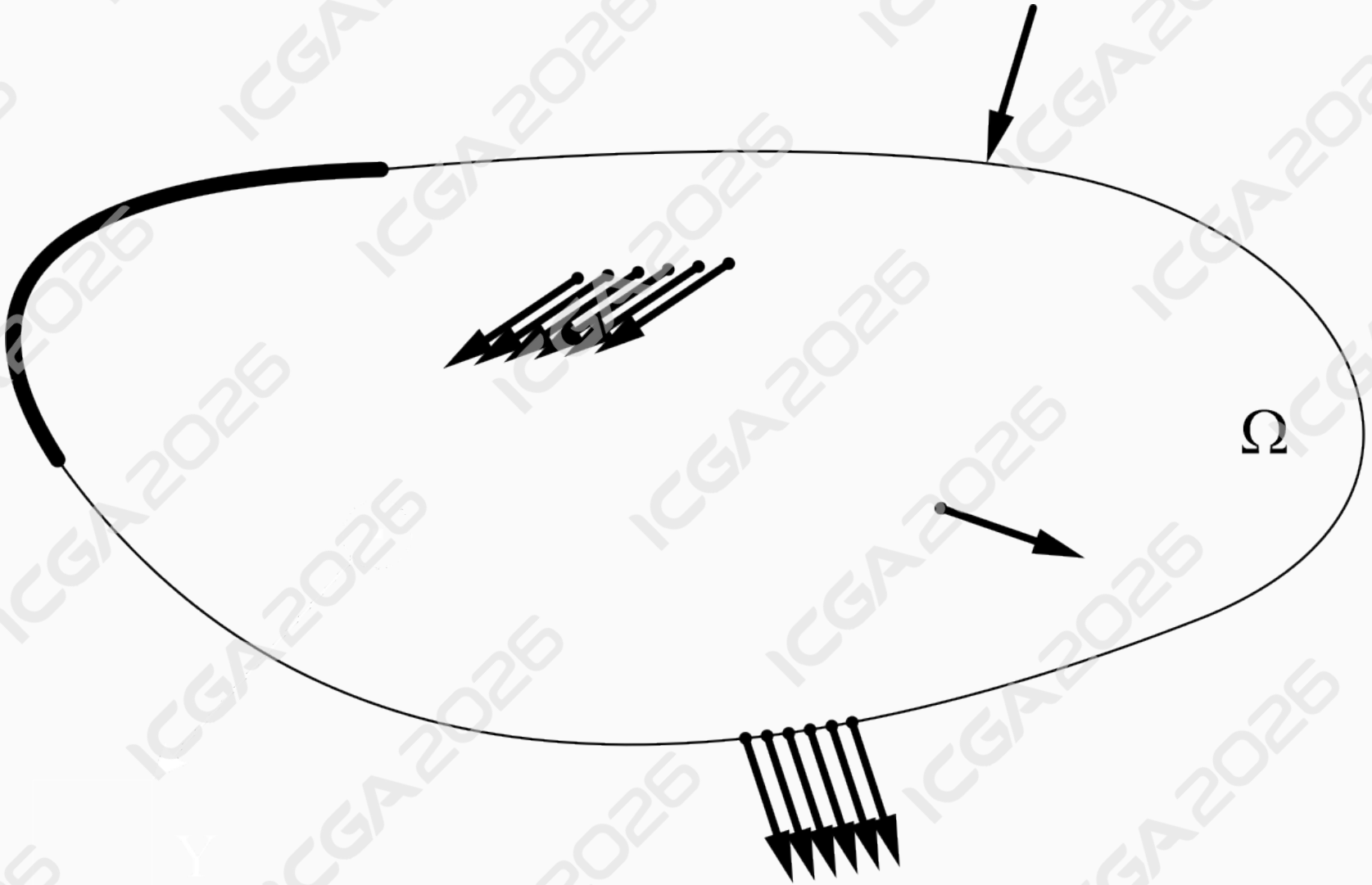
Minimize the weighted sum of compliance and material ‘cost’

$$\inf_{\sigma,\eta} \inf_{\chi} \int_{\Omega} \Big(W(\sigma,\eta) + \ell\chi \Big) d\Omega$$

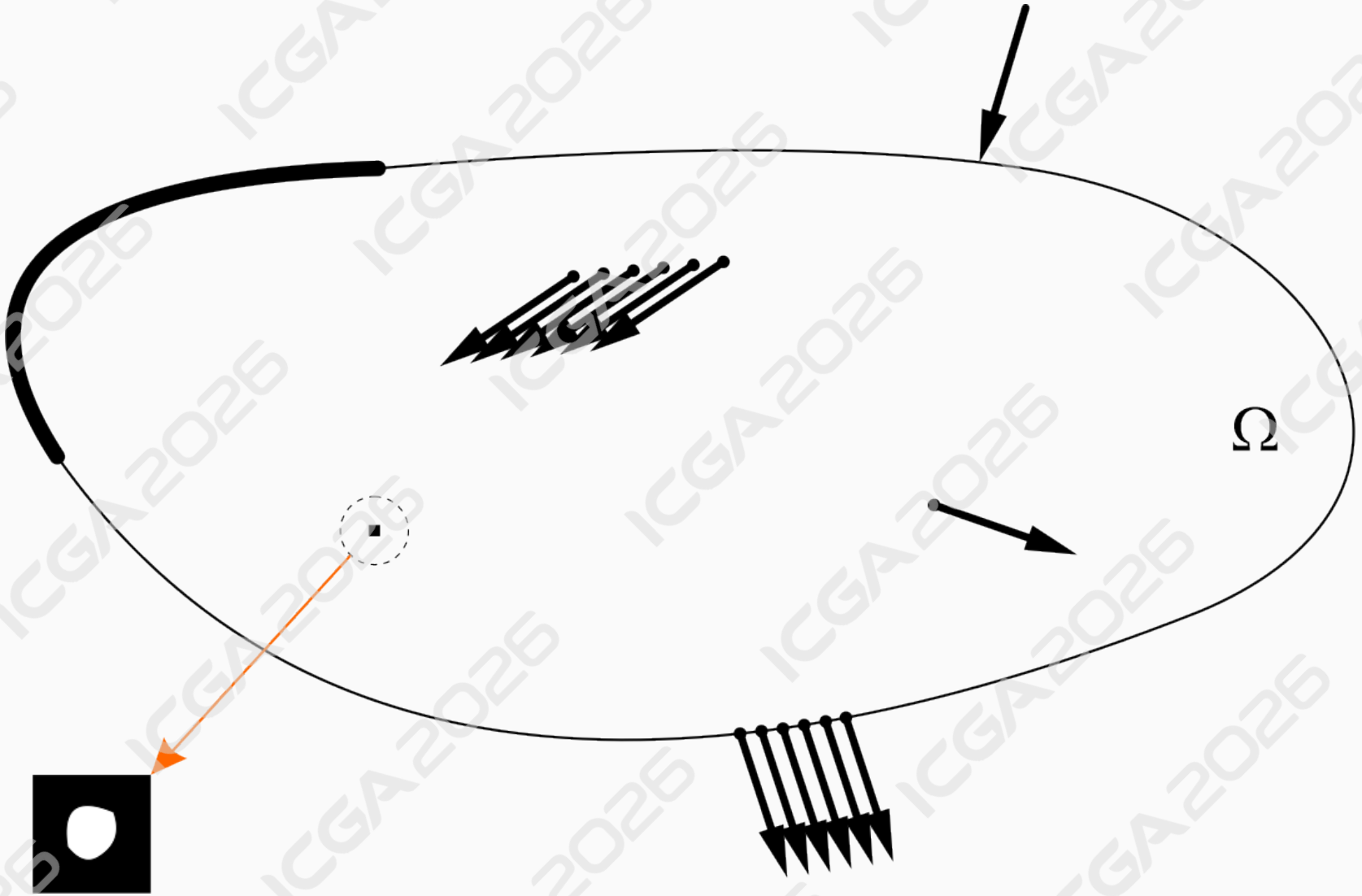
where

W	stress energy function
$\ell > 0$	Lagrange multiplier for the ‘cost’

Optimum design problem – original formulation

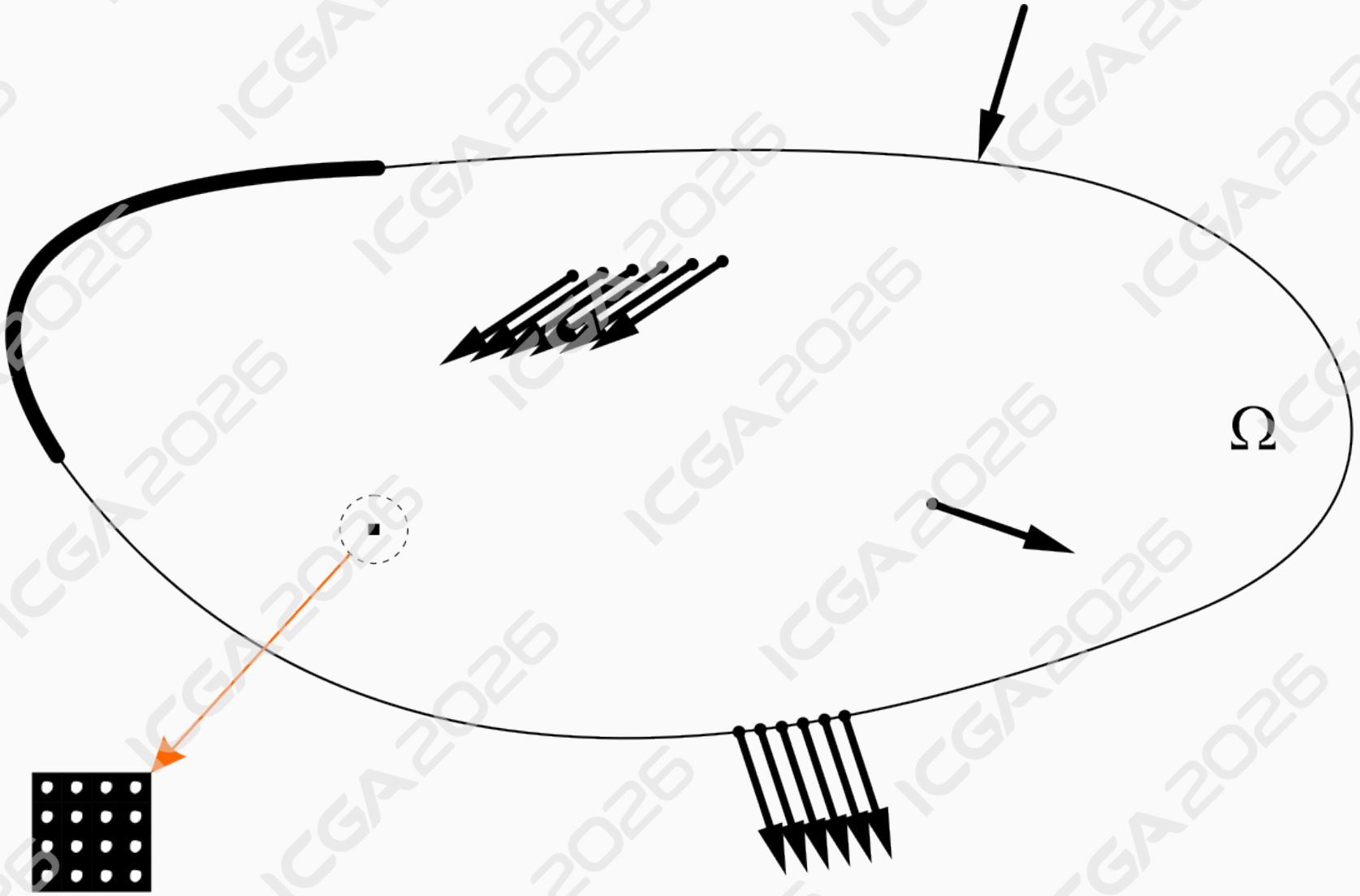


Optimum design problem – original formulation



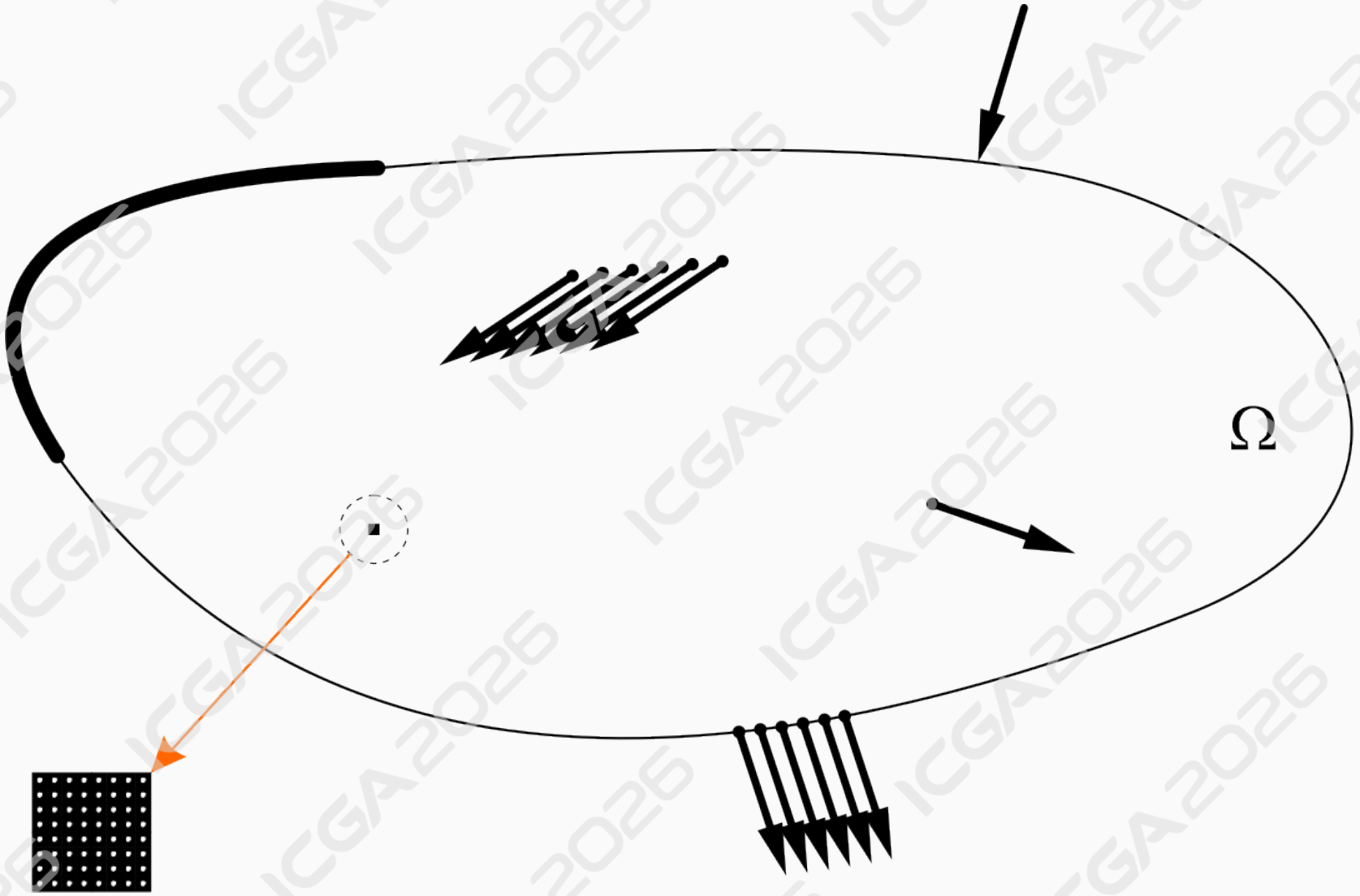
Stress energy function W decreases
when the precision of material distribution increases

Optimum design problem – original formulation



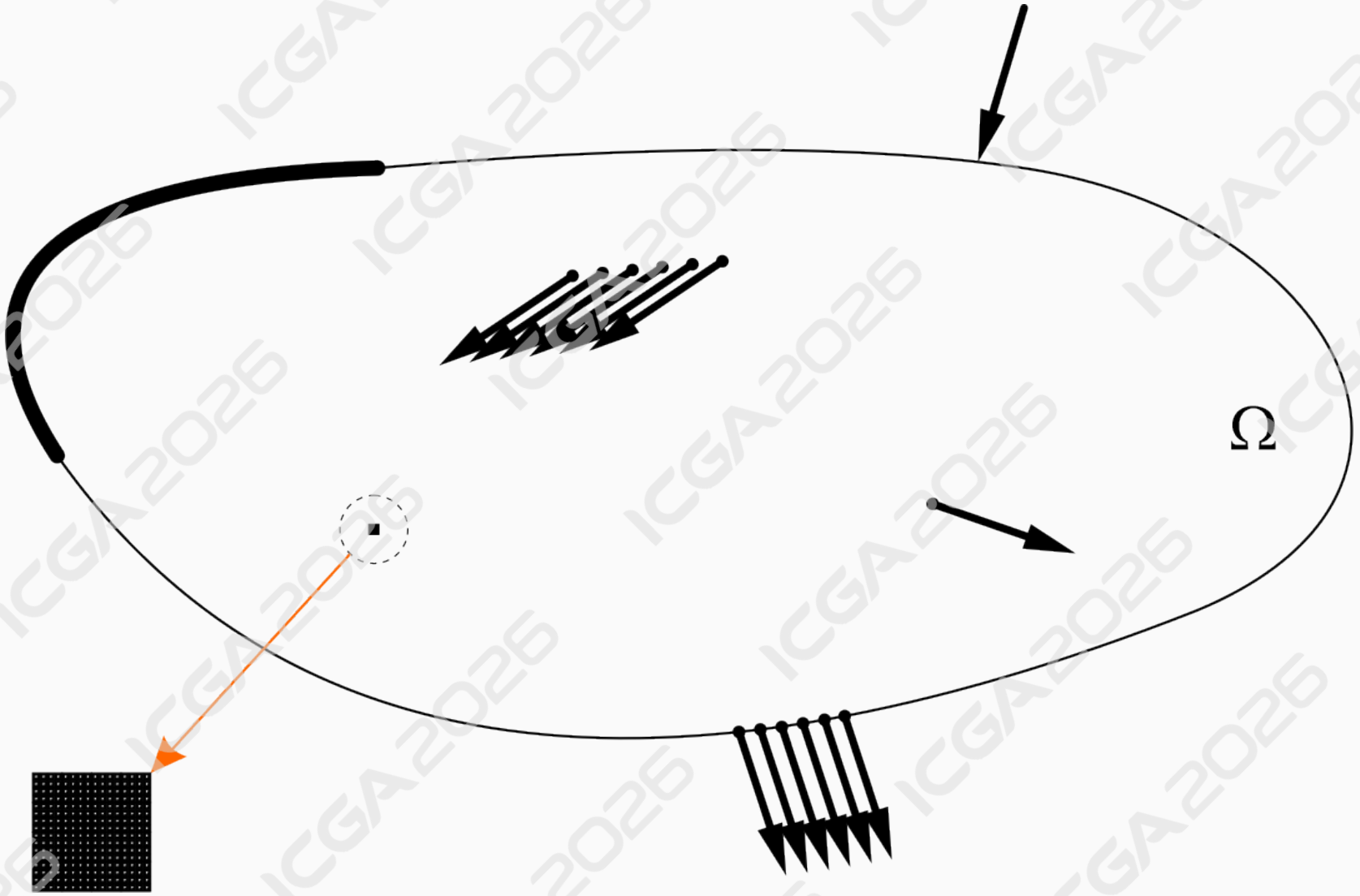
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Optimum design problem – original formulation



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Optimum design problem – original formulation



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Optimum design problem

(relaxed formulation)

Optimum design problem – relaxed formulation

1.

Consider,

- arbitrary $x \in \Omega$ and its neighborhood Y (microstructure)

Optimum design problem – relaxed formulation

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- $f = \langle \phi \rangle$ pointwise in Ω

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- $N = \langle \sigma \rangle$ membrane forces in Ω, $N(x) \in \mathcal{M}^{2\times 2}_{sym}$
- $M = \langle \eta \rangle$ bending moments in Ω, $M(x) \in \mathcal{M}^{2\times 2}_{sym}$

Optimum design problem – relaxed formulation

2.

Introduce QW – the quasiconvex envelope of W

$$QW(\theta, N, M) = \min_{\substack{\text{all} \\ \text{possible} \\ \text{microstructures}}} \langle W(\sigma, \eta) \rangle$$

Optimum design problem – relaxed formulation

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How tight is the quasiconvex bound?

$QW = \langle W \rangle$ if it is attained by a microstructure,

but none has been identified yet

for the membrane-and-bending problem

Optimum design problem – relaxed formulation

3.

Explicit formula for *QW*

$$QW(\theta, N, M) \quad = \quad W(N, M) + \frac{1-\theta}{\theta}\left(W(N, M) + H(N, M)\right)$$

Optimum design problem – relaxed formulation

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Explicit formula for QW

$$QW(\theta, N, M) = W(N, M) + \frac{1-\theta}{\theta}\left(W(N, M) + H(N, M)\right)$$

$$W(N, M) = \frac{K+L}{4}\left((\operatorname{tr} N)^2 + (\operatorname{tr} M)^2\right) - L\left(\det N + \det M\right)$$

K, L coefficients of 2D isotropy

Optimum design problem – relaxed formulation

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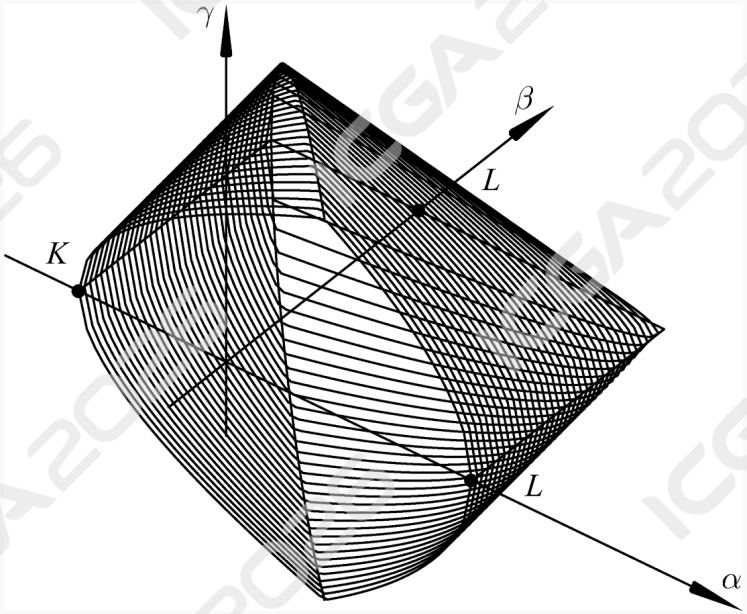
K, L coefficients of 2D isotropy

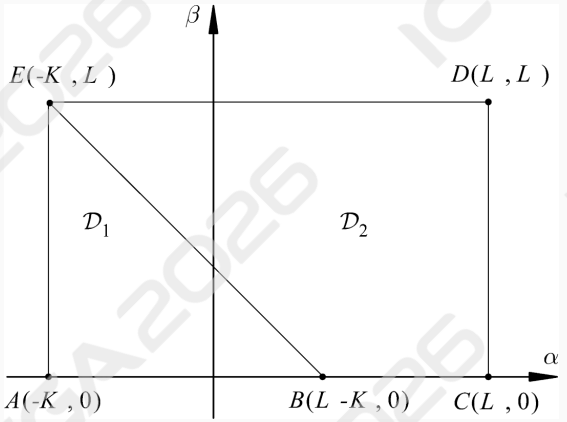
$$H(N, M) = \max_{\alpha,\beta,\gamma}\{\alpha \det N + \beta \det M + \gamma\left(\operatorname{tr} N \operatorname{tr} M - \operatorname{tr} NM\right)\}$$

α, β, γ real parameters

Optimum design problem – relaxed formulation

Explicit formula for *QW*

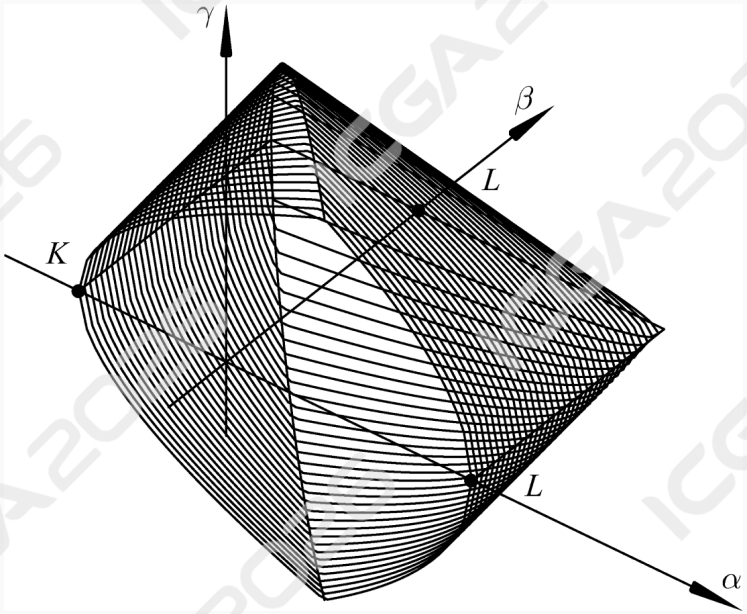


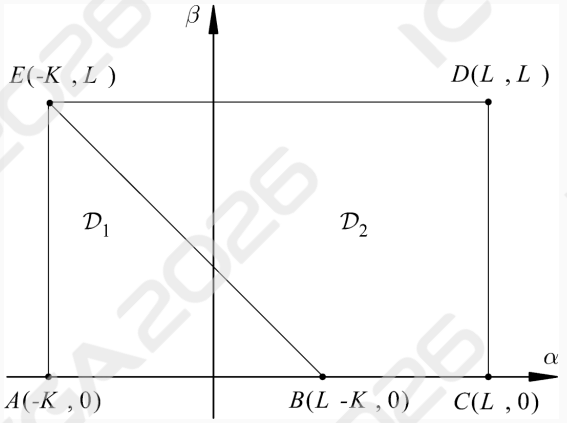


$-K \leq \alpha \leq L$ (as in the pure membrane problem)

Optimum design problem – relaxed formulation

Explicit formula for *QW*



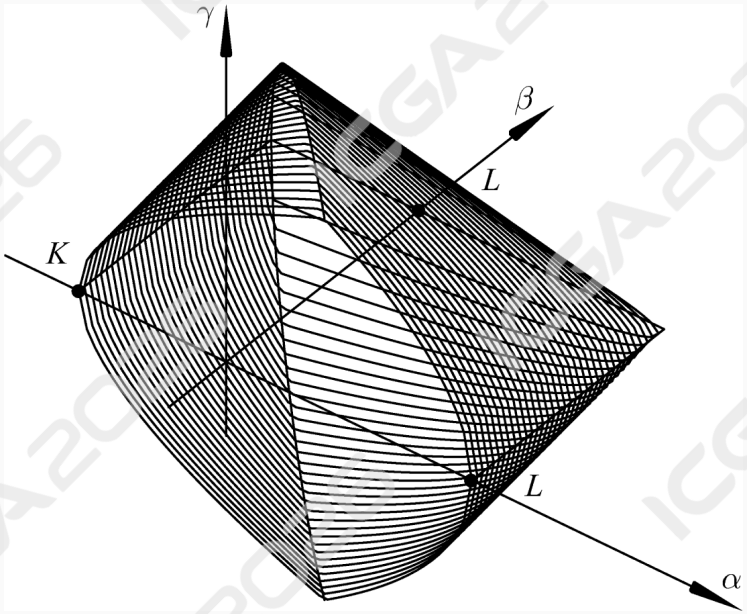


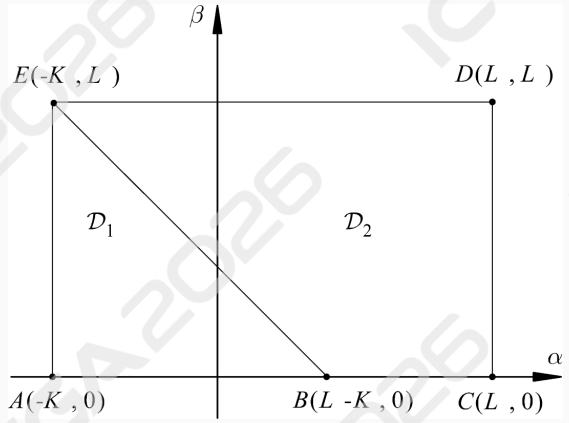
$-K \leq \alpha \leq L$ (as in the pure membrane problem)

$0 \leq \beta \leq L$ (as in the pure bending problem)

Optimum design problem – relaxed formulation

Explicit formula for *QW*





$$-K \le \alpha \le L \qquad \text{(as in the pure membrane problem)}$$

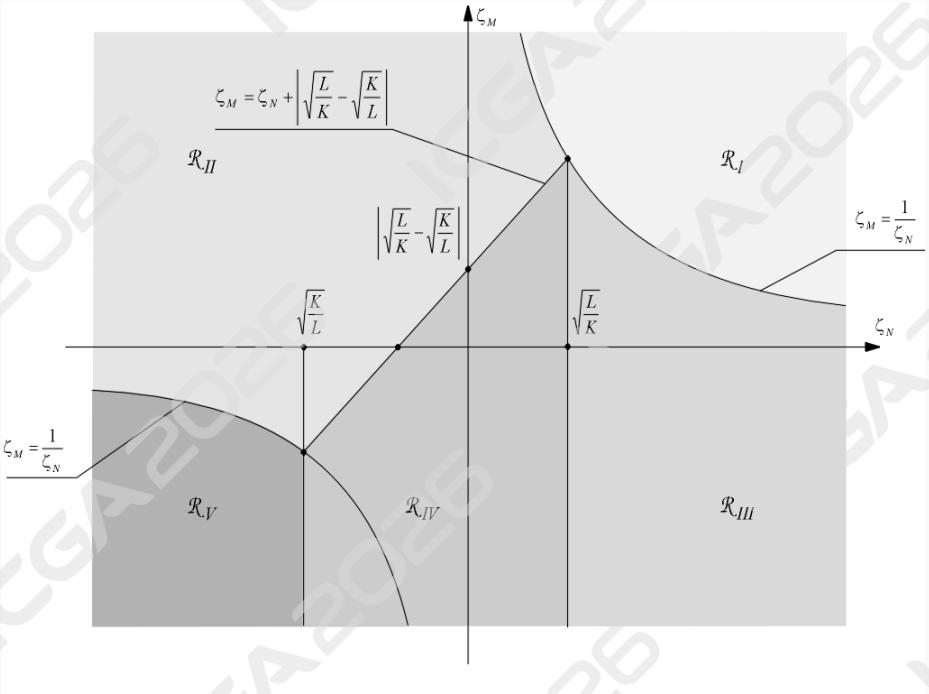
$$0 \le \beta \le L \qquad \text{(as in the pure bending problem)}$$

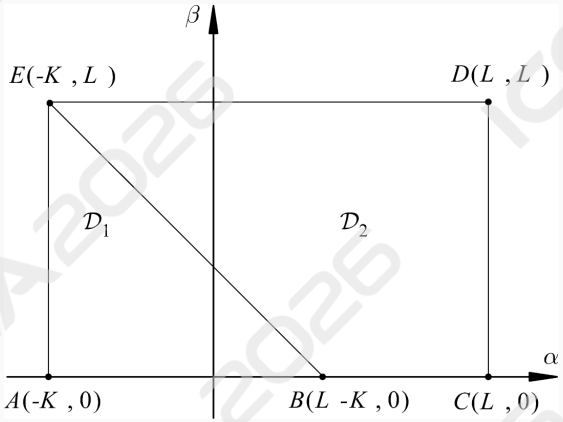
$$\gamma^2 \le (K+\alpha)(K+\beta) \qquad \text{if} \quad (\alpha,\beta) \in \mathcal{D}_1$$

$$\gamma^2 \le (L-\alpha)(L-\beta) \qquad \text{if} \quad (\alpha,\beta) \in \mathcal{D}_2$$

Optimum design problem – relaxed formulation

Explicit formula for *QW*

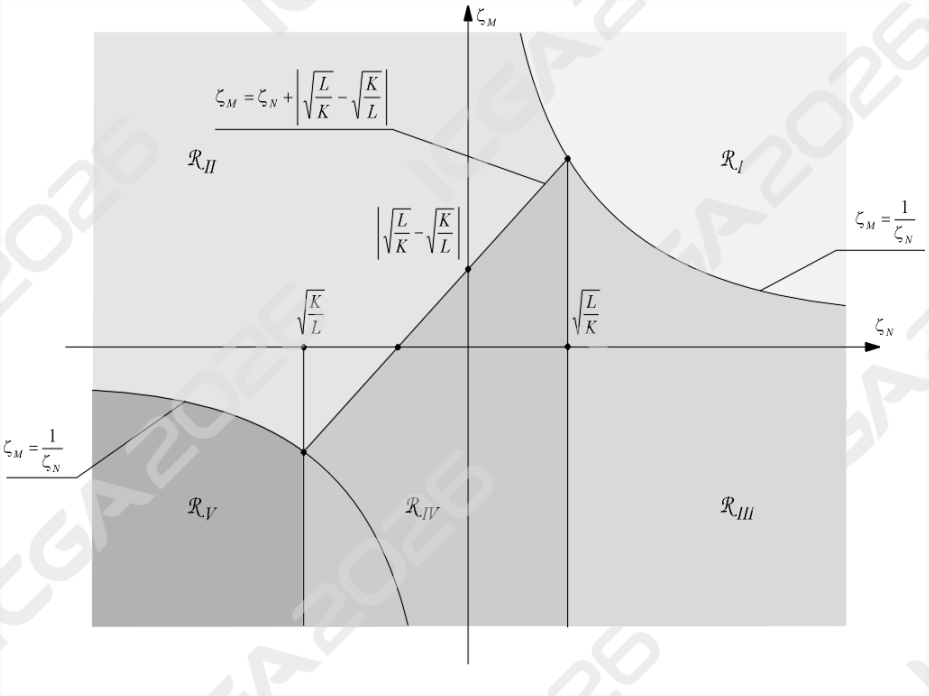


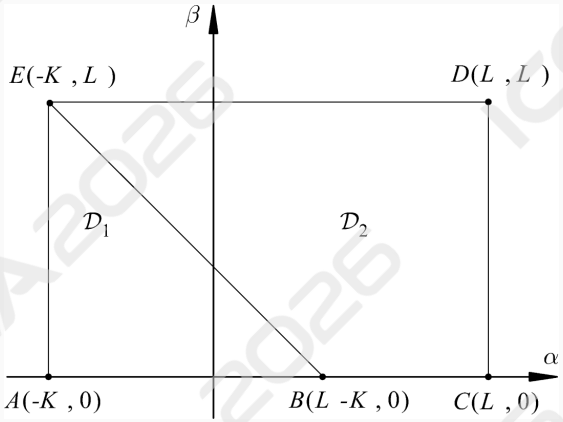


Regime $\mathcal{R}_I$ corresponds to the point *D*

Optimum design problem – relaxed formulation

Explicit formula for *QW*



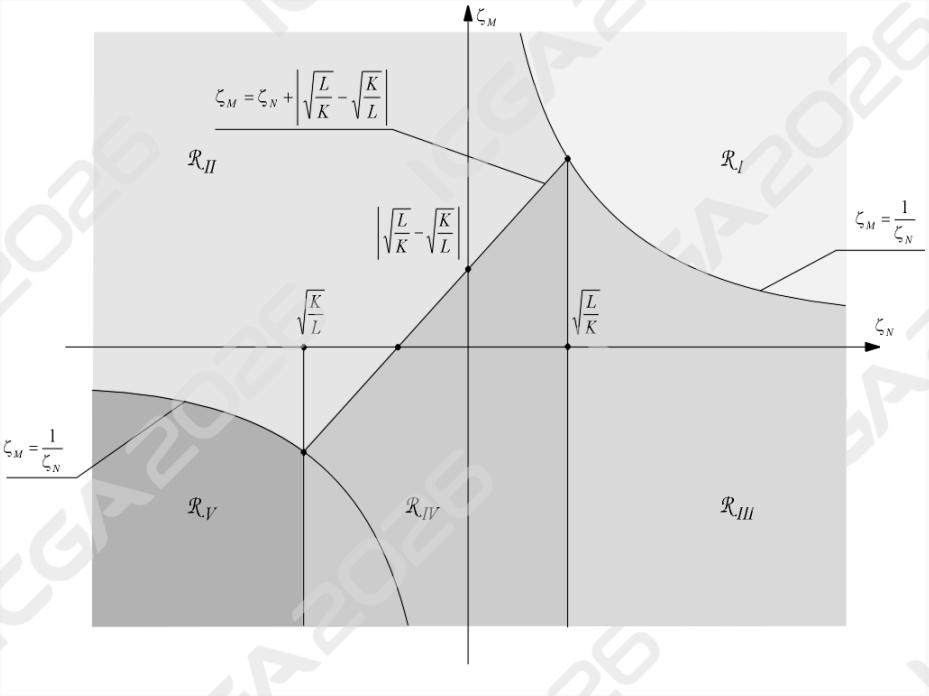


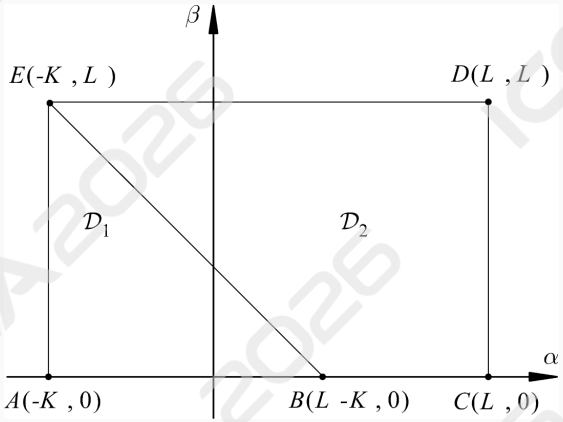
Regime $\mathcal{R}_I$ corresponds to the point *D*

Regime $\mathcal{R}_{II}$ corresponds to the segment *BE*

Optimum design problem – relaxed formulation

Explicit formula for *QW*





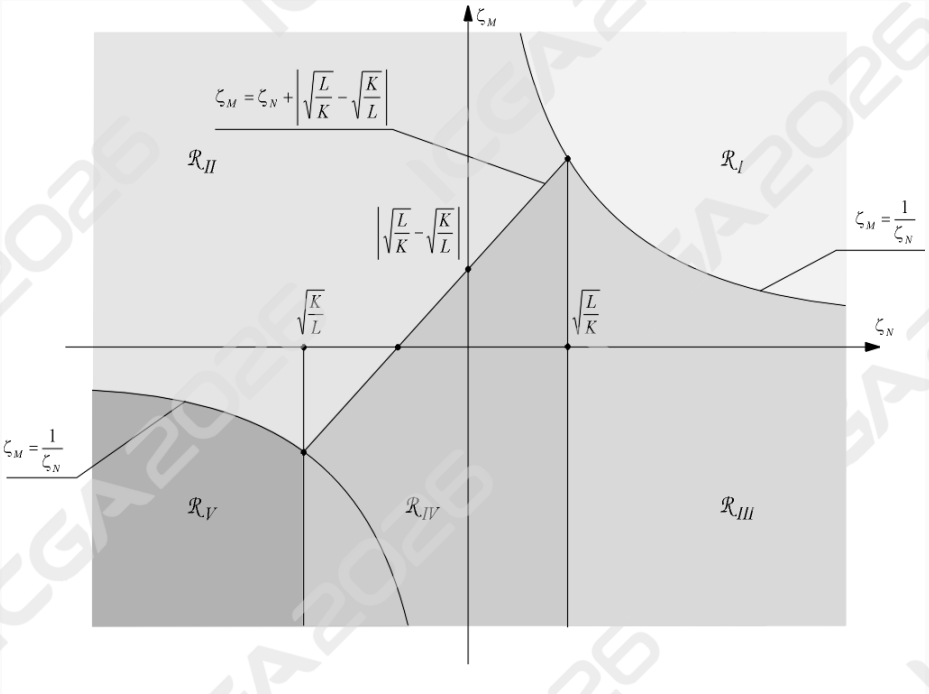
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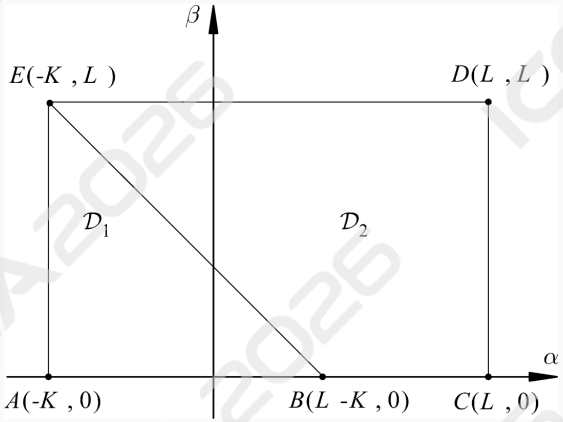
Regime $\mathcal{R}_{II}$ corresponds to the segment *BE*

Regime $\mathcal{R}_{III}$ corresponds to the segment *BC*

Optimum design problem – relaxed formulation

Explicit formula for *QW*





Regime $\mathcal{R}_I$ corresponds to the point *D*

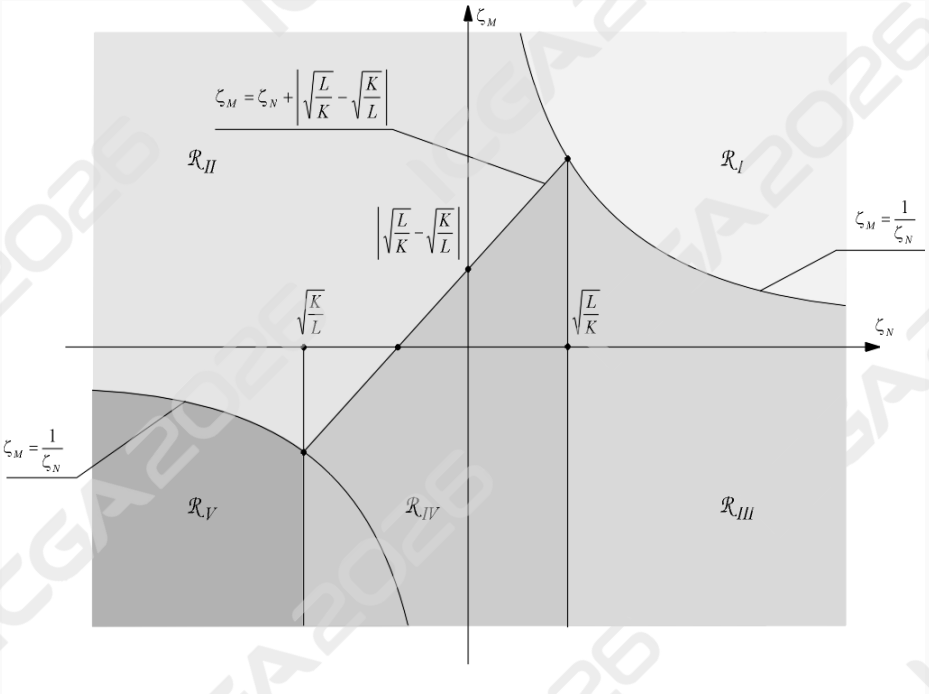
Regime $\mathcal{R}_{II}$ corresponds to the segment *BE*

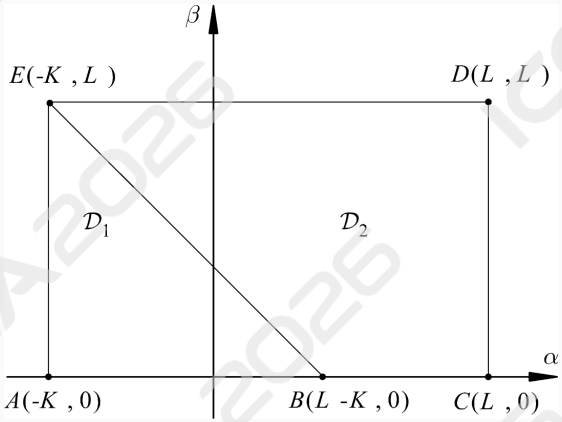
Regime $\mathcal{R}_{III}$ corresponds to the segment *BC*

Regime $\mathcal{R}_{IV}$ corresponds to the point *B*

Optimum design problem – relaxed formulation

Explicit formula for *QW*





Regime $\mathcal{R}_I$ corresponds to the point *D*

Regime $\mathcal{R}_{II}$ corresponds to the segment *BE*

Regime $\mathcal{R}_{III}$ corresponds to the segment *BC*

Regime $\mathcal{R}_{IV}$ corresponds to the point *B*

Regime $\mathcal{R}_V$ corresponds to the segment *AB*

Optimum design problem – relaxed formulation

4.

Determining the optimum θ

$$\min_{\theta \in [0,1]} \Big(QW(\theta, N, M) + \ell\theta \Big)$$

Optimum design problem – relaxed formulation

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Determining the optimum θ

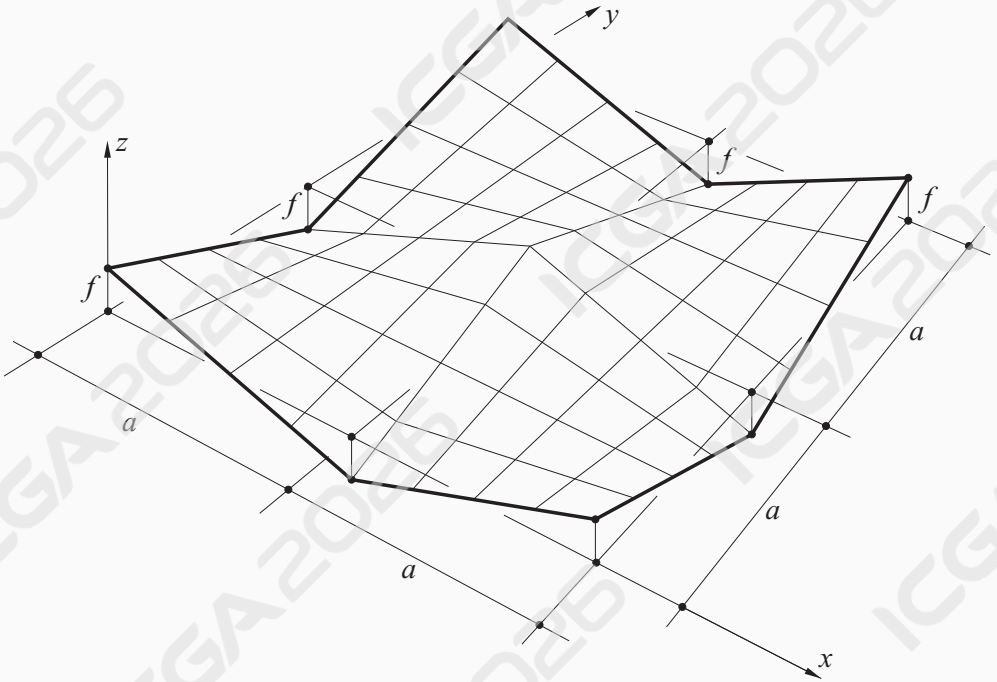
$$\min_{\theta \in [0,1]} \Big(QW(\theta, N, M) + \ell\theta \Big)$$

gives

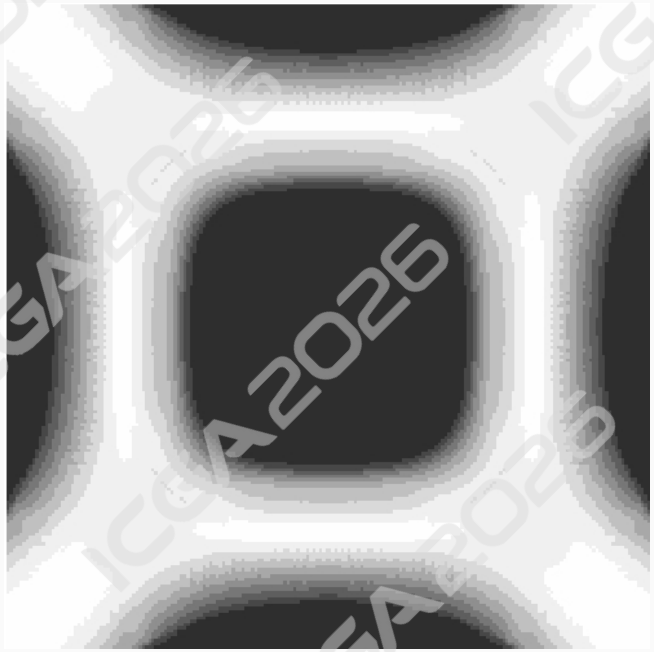
$$\theta_{\text{opt}}(N, M) = \min \left\{ 1, \sqrt{\frac{W(N, M) + H(N, M)}{\ell}} \right\}$$

Example

Example



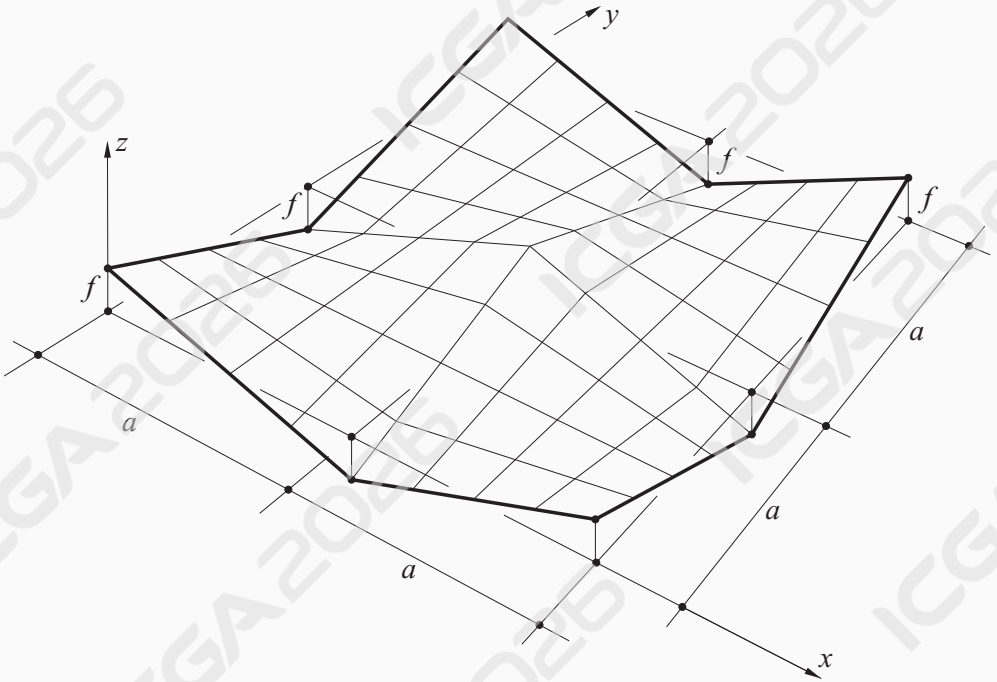
Structure clamped along the contour and loaded uniformly in the z-direction



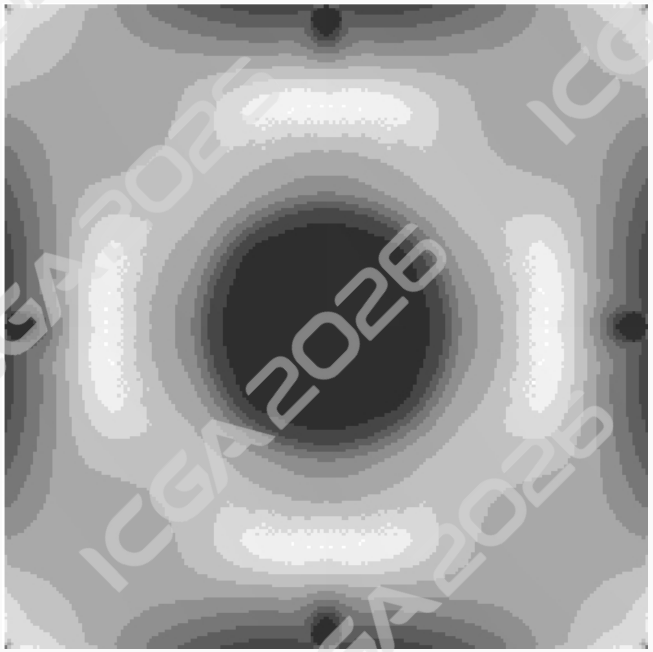
Material layout for $f = 0$ (flat plate)

Grayscale represents material fraction (white=void, black=solid)

Example



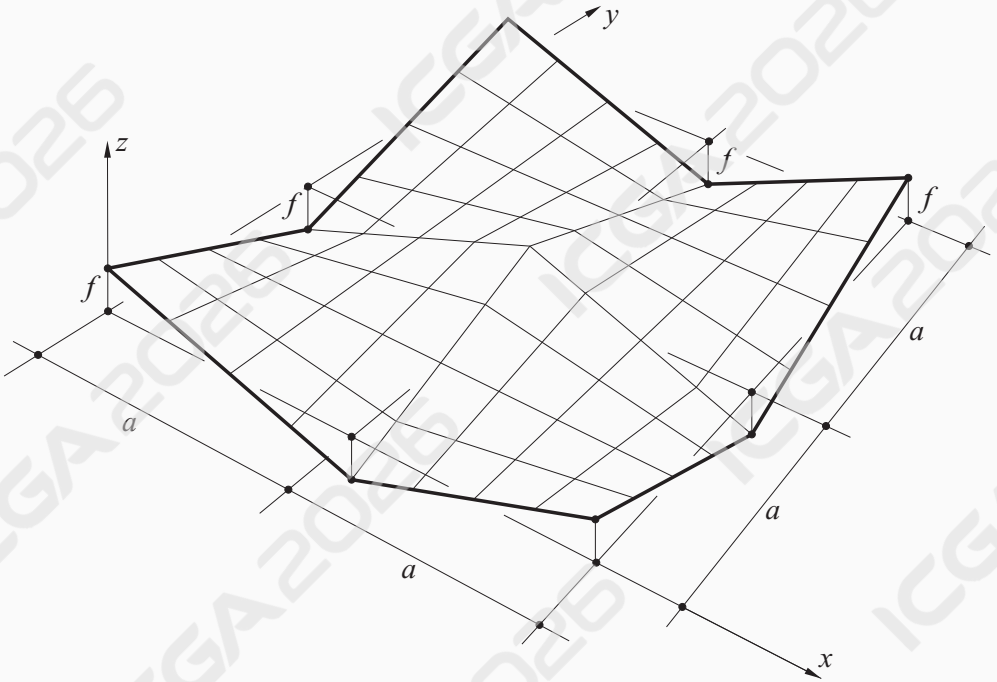
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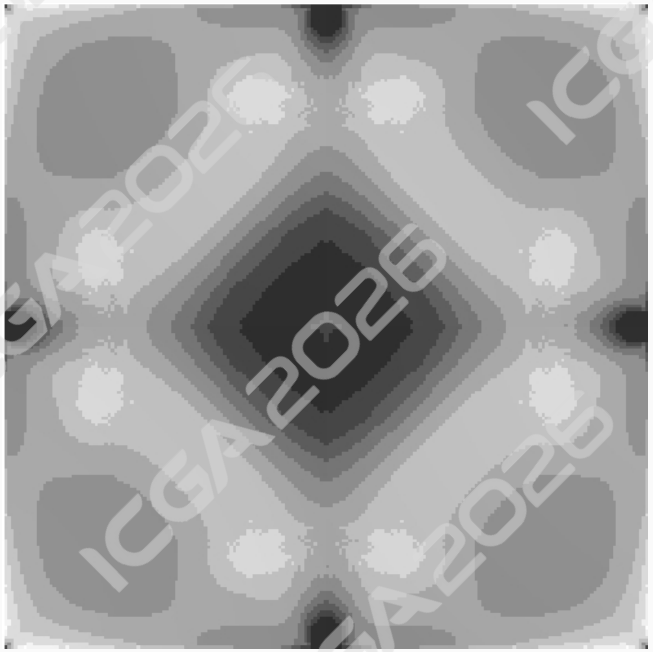
Material layout for $f/a = 1/20$

Grayscale represents material fraction (white=void, black=solid)

Example



Structure clamped along the contour and loaded uniformly in the z-direction



Material layout for $f/a = 1/10$

Grayscale represents material fraction (white=void, black=solid)

Key takeaways

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Versatility: The method also handles two non-degenerate isotropic materials

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Drawback: Microstructures are not yet known

Thank you!