# Stability of equilibrium of straight and slender elastic bars of atypical cross-sections: a combined axial, lateral, and torsional buckling analysis

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# A unified approach to stability of spatial elastic bodies

The domain of the body $\Omega$ is parameterized by the Cartesian orthogonal system $(x_1, x_2, x_3)$ and $d\Omega = dx_1 dx_2 dx_3$.
The initial loads are proportional to a multiplier $\lambda$. The initial stresses are: $\lambda\sigma_0^{ij}$.
The body is made of material with the anisotropic moduli $C^{ijkl}(\boldsymbol{x})$
The components of displacement $\boldsymbol{u}$ will be denoted by $u_1$, $u_2$, $u_3$.
Partial derivative $()_{,i} = \partial/\partial x_i$.
Strains are given by:

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

# A unified approach to stability of spatial elastic bodies

Let $V$ be a set of kinematically admissible displacements.
According to linear theory of stability, Washizu 1982 sec. 5.2:
*Find $\boldsymbol{u} \in V$ such that the multiplier $\lambda$ exists that:*

$$a(\boldsymbol{u}, \boldsymbol{v}) + \lambda b(\boldsymbol{u}, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \text{ kin. adm.} \tag{1}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \sum_{i,j,k,l=1}^{3} C^{ijkl} \varepsilon_{kl}(\boldsymbol{u}) \varepsilon_{ij}(\boldsymbol{v}) d\Omega$$

$$b(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \sum_{i,j,k=1}^{3} \sigma_0^{ij} u_{k,i} v_{k,j} d\Omega$$

The solution of (1) are the pairs: $(\lambda_n, \boldsymbol{u}_{(n)})$, $n \in \mathbb{N}$ and
$|\lambda_1| \leqslant |\lambda_2| \leqslant |\lambda_3| \leqslant \ldots$

# A unified approach to stability of spatial elastic bodies

We are intrested in $\lambda_1 = \min(\lambda_1', \lambda_2')$, where

$$\frac{1}{\lambda_1'} = \max_{\boldsymbol{v} \text{ kin. adm.}} \frac{W_0(\boldsymbol{v})}{W(\boldsymbol{v})}, \qquad \frac{1}{\lambda_2'} = \max_{\boldsymbol{v} \text{ kin. adm.}} -\frac{W_0(\boldsymbol{v})}{W(\boldsymbol{v})}$$

or

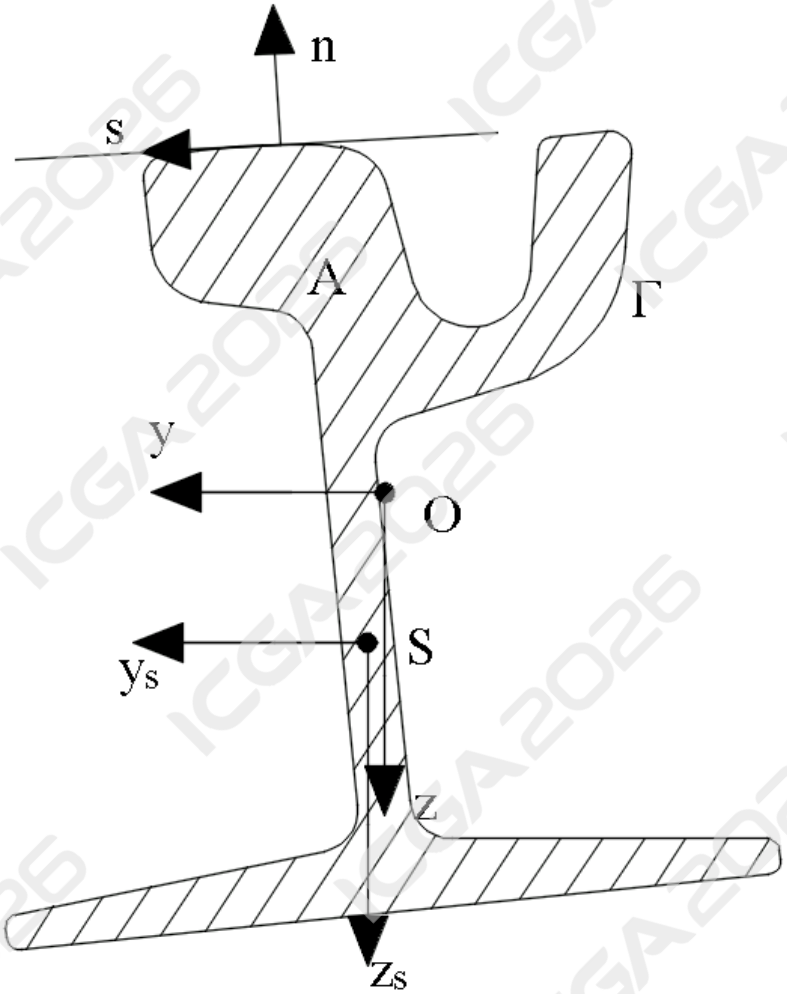
$$\frac{1}{\lambda_1} = \max_{\boldsymbol{v} \text{ kin. adm.}} \pm\frac{W_0(\boldsymbol{v})}{W(\boldsymbol{v})} \tag{2}$$

where

$$W(\boldsymbol{v}) = \frac{1}{2}a(\boldsymbol{v}, \boldsymbol{v}), \quad W_0(\boldsymbol{v}) = \frac{1}{2}b(\boldsymbol{v}, \boldsymbol{v})$$

Formula (2) is called the Rayleigh quotient.

# Stability of straight bars of arbitrary cross sections - Vlasov-like theory




$A$ - area of the cross section
$J_y$ - moment of inertia about y axis
$J_z$ - moment of inertia about z axis
$J$ - torsional moment of inertia
$I_\omega$ - warping modulus
$O$ - gravity center
$S$ - shear center
$\boldsymbol{n}$ - unit normal vector $||\boldsymbol{n}|| = 1$

# Stability of straight bars of arbitrary cross sections - Vlasov-like theory

We consider: a straight, prismatic bar made of elastic isotropic homogeneous material.
The initial loads being proportional to a multiplier $\lambda$ cause the internal forces:

$$\lambda N^0(x),\ \lambda M_z^0(x),\ \lambda M_y^0(x),\ \lambda T_z^0(x),\ \lambda T_y^0(x),\ \lambda \mathcal{M}^0(x),\ \lambda B_x^0(x)$$

In Vlasov-like theory, the three non-zero stress components $\sigma_0^{11}$, $\sigma_0^{12}$, $\sigma_0^{13}$ are expressed by:

$$\sigma_0^{11} = \frac{1}{A}N^0(x) - \frac{1}{J_z}yM_z^0(x) + \frac{1}{J_y}zM_y^0(x) + \frac{1}{I_\omega}\omega(y,z)B_x^0(x)$$

$$\sigma_0^{12} = \frac{1}{J}\left(\frac{\partial\omega}{\partial y} - z + z_s\right)\mathcal{M}^0(x) + \frac{1}{A}\frac{\partial\zeta}{\partial y}\left(\alpha_z T_z^0(x) + \alpha_{zy}T_y^0(x)\right) + \frac{1}{A}\frac{\partial\eta}{\partial y}\left(\alpha_{yz}T_z^0(x) + \alpha_y T_y^0(x)\right),$$

$$\sigma_0^{13} = \frac{1}{J}\left(\frac{\partial\omega}{\partial z} + y - y_s\right)\mathcal{M}^0(x) + \frac{1}{A}\frac{\partial\zeta}{\partial z}\left(\alpha_z T_z^0(x) + \alpha_{zy}T_y^0(x)\right) + \frac{1}{A}\frac{\partial\eta}{\partial z}\left(\alpha_{yz}T_z^0(x) + \alpha_y T_y^0(x)\right),$$

# Warping functions

Kinematic assumptions:

$$u_x = u(x) + y\varphi(x) + z\beta(x) + \omega(y,z)\frac{d\theta}{dx} + [\zeta(y,z) - z]\left(\beta + \frac{dw}{dx}\right) + [\eta(y,z) - y]\left(\varphi + \frac{dv}{dx}\right)$$

$$u_y = v(x) - (z - z_s)\theta(x)$$

$$u_z = w(x) + (y - y_s)\theta(x)$$

Independent displacements fields: $u$, $v$, $v$, $\beta$, $\varphi$, $\theta$.
The warping functions:

| | |
|---|---|
| $\omega(y,z)$ | – warping due to torsion |
| $\zeta(y,z) - z$ | – warping due to transverse shear in $z$ direction |
| $\eta(y,z) - y$ | – warping due to transverse shear in $y$ direction |

Lewiński T., Czarnecki, S., On incorporating warping effects due to transverse shear and torsion into the theories of straight elastic bars, Acta Mech 232(1), 247–282 (2021). https://doi.org/10.1007/s00707-020-02849-7

# Warping function due to torsion

The warping function $\omega(y,z)$ is expressed by:

$$\omega(y,z) = \omega_0(y,z) + y_s z - z_s y$$

where $\omega_0(y,z)$ is the solution of the elliptic problem:

$$\Delta\omega_0 = 0 \quad \text{in} \quad \mathcal{A}$$

$$\frac{\partial\omega_0}{\partial \boldsymbol{n}} = n_y z - n_z y \quad \text{on} \quad \Gamma$$

$$\int_{\mathcal{A}} \omega_0 d\mathcal{A} = 0$$

The position of point $S = (y_s, z_s)$ is given by:

$$y_s = -\frac{1}{J_y}\int_{\mathcal{A}} z\omega_0 d\mathcal{A}, \quad z_s = \frac{1}{J_z}\int_{\mathcal{A}} y\omega_0 d\mathcal{A}$$

# Warping functions due to transverse shear

Notation: $(f|g) = \int_A f g d\mathcal{A}$.

Warping functions due to transverse shear $\eta = \eta(y,z)$, $\zeta = \zeta(y,z)$ are given by:

$$\eta(y,z) = \frac{J_z}{D}\left[(z|\hat{\zeta})\hat{\eta}(y,z) - (z|\hat{\eta})\hat{\zeta}(y,z)\right]$$

$$\zeta(y,z) = \frac{J_y}{D}\left[(y|\hat{\eta})\hat{\zeta}(y,z) - (y|\hat{\zeta})\hat{\eta}(y,z)\right]$$

where $D = (\hat{\eta}|y)(\hat{\zeta}|z) - (\hat{\eta}|z)(\hat{\zeta}|y)$ and auxiliary functions $\hat{\zeta} = \hat{\zeta}(y,z)$, $\hat{\eta} = \hat{\eta}(y,z)$ are the solutions of the elliptic problems:

$$\Delta\hat{\zeta} + z = 0 \text{ in } \mathcal{A}; \quad \frac{\partial\hat{\zeta}}{\partial n} = 0 \text{ on } \Gamma, \int_{\mathcal{A}} \hat{\zeta} d\mathcal{A} = 0$$

$$\Delta\hat{\eta} + y = 0 \text{ in } \mathcal{A}; \quad \frac{\partial\hat{\eta}}{\partial n} = 0 \text{ on } \Gamma, \int_{\mathcal{A}} \hat{\eta} d\mathcal{A} = 0$$

Moreover: $(\hat{\eta}|z) = (\hat{\zeta}|y)$.

# Numerical Example - GCRD42 rail

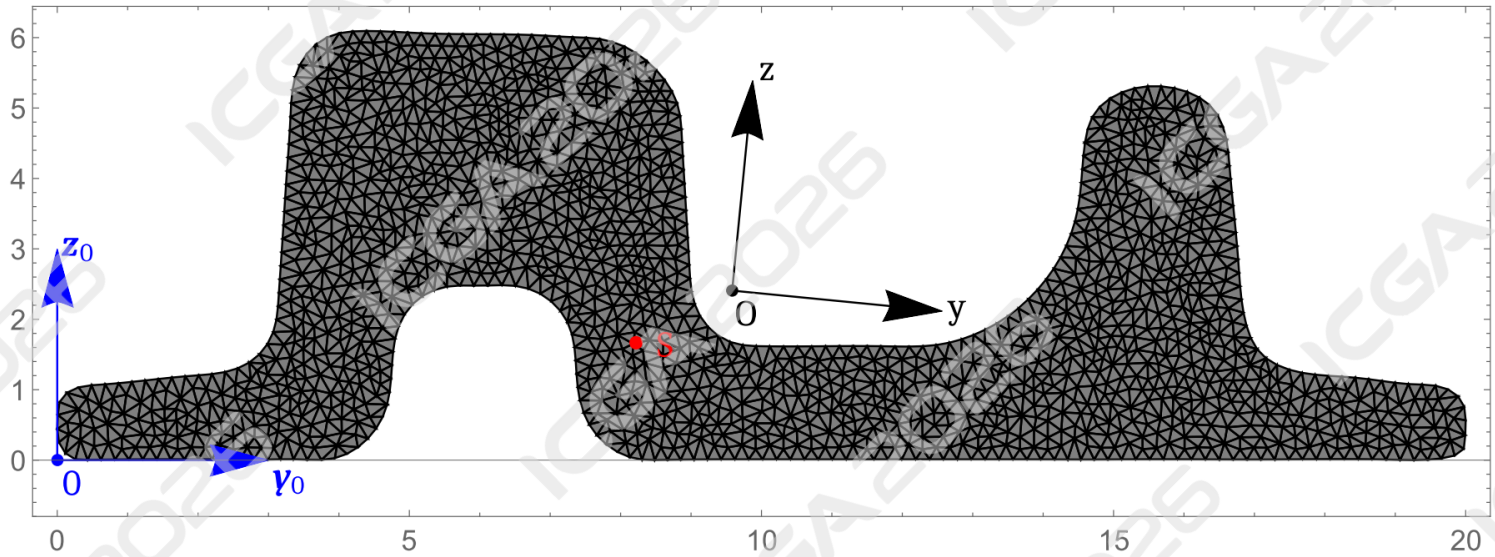


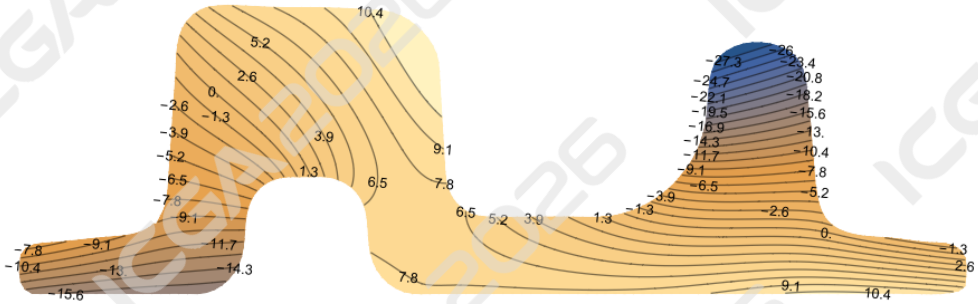

**Fig. 5**: Profile of GCRD42 Girder rail. The initial coordinate system $y_0$, $z_0$ is drawn in blue. The principal axes $y$, $z$ are drawn in black. The red dot indicates point S

**Table 1**: Characteristics of the cross-sections shown in Fig.5; $y_{0(O)}$, $z_{0(O)}$ are the coordinates of the centroid given in the initial coordinate system; $y_{0(S)}$, $z_{0(S)}$ are the coordinates of the point S in the initial coordinate system

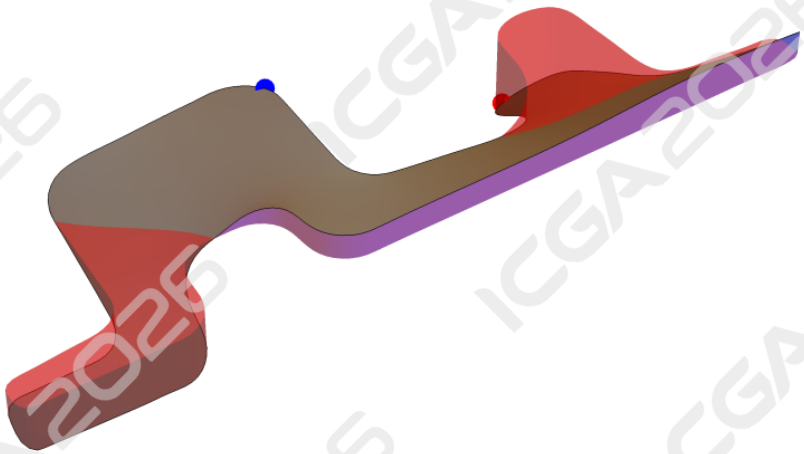
| $A$ $[cm^2]$ | $J_y$ $[cm^4]$ | $J_z$ $[cm^4]$ | $J$ $[cm^4]$ | $y_{0(O)}$ $[cm]$ | $z_{0(O)}$ $[cm]$ | $y_{0(S)}$ $[cm]$ | $z_{0(S)}$ $[cm]$ | $\measuredangle(y_0, y)$ $[°]$ | $k_y$ $[-]$ | $k_{yz}{=}k_{zy}$ $[-]$ | $k_z$ $[-]$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 56.52 | 156.9 | 1471 | 108.8 | 9.58 | 2.41 | 8.22 | 1.66 | -5.58 | 0.452 | 0.0601 | 0.494 |

Czubacki, R., Lewiński, T. The six-parameter theory of straight elastic bars with transverse shear–torsion decoupling. Acta Mech 237, 2435–2480 (2026). https://doi.org/10.1007/s00707-025-04606-0

# GCRD42 rail - Warping function due to torsion




(a) contour plot of $\omega$ warping function

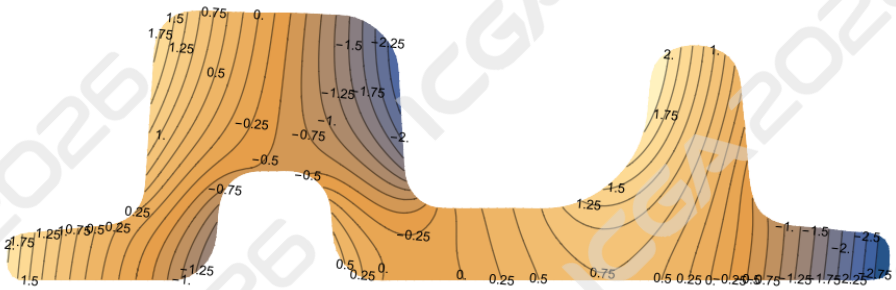


(b) Red indicates negative values, and blue indicates positive values. The blue and red dots indicate minimal and maximal values, respectively

**Fig. 6**: Plots of warping function $\omega$: contour plot (a), 3D plot (b)

Czubacki, R., Lewiński, T. The six-parameter theory of straight elastic bars with transverse shear–torsion decoupling. Acta Mech 237, 2435–2480 (2026). https://doi.org/10.1007/s00707-025-04606-0

# GCRD42 rail - Warping function due to transverse shear

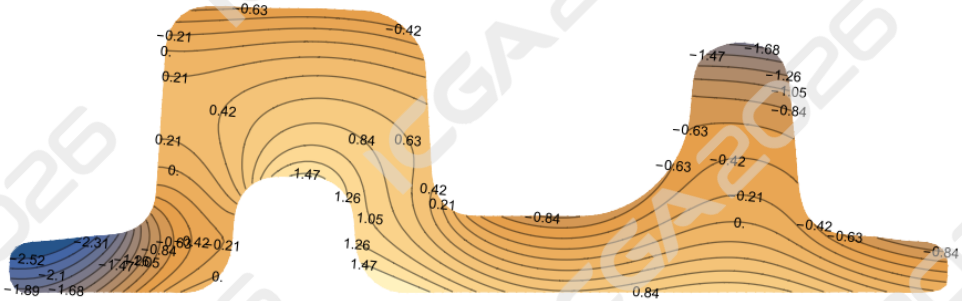


(a) contour plot of $\eta - y$ warping function

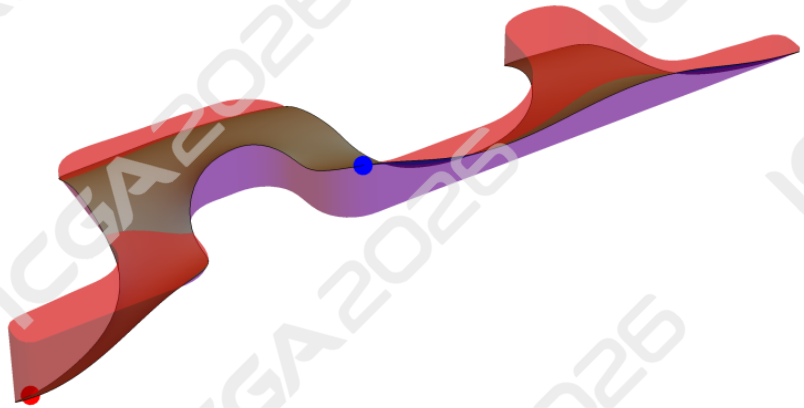


(b) Red indicates negative values, and blue indicates positive values. The blue and red dots indicate minimal and maximal values, respectively

**Fig. 7**: Plots of warping function $\eta - y$: contour plot (a), 3D plot (b)



(a) contour plot of $\zeta - z$ warping function



(b) Red indicates negative values, and blue indicates positive values. The blue and red dots indicate minimal and maximal values, respectively

**Fig. 8**: Plots of warping function $\zeta - z$: contour plot (a), 3D plot (b)

Czubacki, R., Lewiński, T. The six-parameter theory of straight elastic bars with transverse shear–torsion decoupling. Acta Mech 237, 2435–2480 (2026). https://doi.org/10.1007/s00707-025-04606-0

# The shear areas

$$A_z = \frac{(J_y)^2}{D}(\widehat{\eta}|y)$$

$$A_z = \frac{(J_z)^2}{D}(\widehat{\zeta}|z)$$

$$A_{yz} = A_{zy} = -\frac{J_z J_y}{D}(\widehat{\eta}|y) = -\frac{J_z J_y}{D}(\widehat{\zeta}|z)$$

$$\begin{bmatrix} \alpha_z/A & \alpha_{zy}/A \\ \alpha_{yz}/A & \alpha_y/A \end{bmatrix} = \begin{bmatrix} A_z & A_{zy} \\ A_{yz} & A_y \end{bmatrix}^{-1}$$

Quantities $A_y$, $A_{yz} = A_{zy}$, $A_z$ are the reduced areas (the shear areas). We can introduce the *the shear correction factors:*

$$A_z = k_z A, \quad A_{zy} = A_{yz} = k_{yz} A = k_{zy} A, \quad A_y = k_y A$$

# Stability of straigth bars of arbitrary cross sections - Vlasov-like theory

Kinematic assumptions in Vlasov-like theory:

$$u_1 = u(x) - y\frac{dv}{dx} - z\frac{dw}{dx} + \omega(y,z)\frac{d\theta}{dx}$$
$$u_2 = v(x) - (z - z_s)\theta(x)$$
$$u_3 = w(x) + (y - y_s)\theta(x)$$

The displacement gradients

$$u_{1,1} = \frac{du}{dx} - y\frac{d^2v}{dx^2} - z\frac{d^2w}{dx^2} + \omega\frac{d^2\theta}{dx^2}$$
$$u_{1,2} = -\frac{dv}{dx} + \frac{\partial\omega}{\partial y}\frac{d\theta}{dx}$$
$$u_{1,3} = -\frac{dw}{dx} + \frac{\partial\omega}{\partial z}\frac{d\theta}{dx}$$

$$u_{2,1} = \frac{dv}{dx} - (z - z_s)\frac{d\theta}{dx}$$
$$u_{2,2} = 0$$
$$u_{2,3} = -\theta$$

$$u_{3,1} = \frac{dw}{dx} + (y - y_s)\frac{d\theta}{dx}$$
$$u_{3,2} = \theta$$
$$u_{3,3} = 0$$

# The work of initial stresses $W_0(\boldsymbol{u})$

We calculate $W_0(\boldsymbol{u})$

$$W_0(\boldsymbol{u}) = \frac{1}{2}\int_0^l U(\boldsymbol{u})dx, \qquad U(\boldsymbol{u}) = \sum_{k,i,j=1}^{3}\int_{\mathcal{A}} u_{k,i}\sigma_0^{ij}u_{k,j}d\mathcal{A}$$

$$\sum_{k,i,j=1}^{3} u_{k,i}\sigma_0^{ij}u_{k,j} = \sigma_0^{11}[(u_{1,1})^2 + (u_{2,1})^2 + (u_{3,1})^2] +$$
$$+ 2\sigma_0^{12}(u_{1,1}u_{1,2} + u_{2,2}u_{2,1} + u_{3,2}u_{3,1}) +$$
$$+ 2\sigma_0^{13}(u_{1,1}u_{1,3} + u_{2,1}u_{2,3} + u_{3,1}u_{3,2})$$

$\sigma_0^{22} = 0$, $\sigma_0^{23} = 0$, $\sigma_0^{33} = 0$.
We substitute representations of $\sigma_0^{11}$, $\sigma_0^{12}$, $\sigma_0^{13}$, $u_{2,1}, u_{3,1}, u_{2,2}, u_{2,1}, u_{3,2}, u_{2,3}$ and integrate over $\mathcal{A}$.

# The work of initial stresses $W_0(\boldsymbol{u})$

$$W_0 = \frac{1}{2}\int_0^l [N^0(x)b_N(x) + M_y^0(x)b_y^M(x) + M_z^0(x)b_z^M(x) + B_x^0(x)b_\omega(x) + \mathcal{M}^0(x)b_{\mathcal{M}}(x) + T_z^0(x)b_z^T(x) + T_y^0(x)b_y^T(x)]\,dx + \widetilde{W}_y + \widetilde{W}_z$$

where

$$\widetilde{W}_y = -\frac{a_z}{4}\int_0^l q_y(x)\theta^2(x)dx, \quad \widetilde{W}_z = -\frac{a_y}{4}\int_0^l q_z(x)\theta^2(x)dx$$

$$b_N(x) = \left(\frac{dw}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + 2z_s\frac{dv}{dx}\frac{d\theta}{dx} - 2y_s\frac{dw}{dx}\frac{d\theta}{dx} + (r_0)^2\left(\frac{d\theta}{dx}\right)^2$$

$$b_y^M(x) = a_y\left(\frac{d\theta}{dx}\right)^2 - 2\frac{dv}{dx}\frac{d\theta}{dx}, \quad b_z^M(x) = -a_z\left(\frac{d\theta}{dx}\right)^2 - 2\frac{dw}{dx}\frac{d\theta}{dx}$$

$$b_\omega(x) = \alpha_x\left(\frac{d\theta}{dx}\right)^2, \quad b_{\mathcal{M}}(x) = \left(-\frac{d^2w}{dx^2}\frac{dv}{dx} + \frac{d^2v}{dx^2}\frac{dw}{dx}\right)$$

$$b_y^T(x) = a_z\theta\frac{d\theta}{dx} + 2\theta\frac{dw}{dx}, \quad b_z^T(x) = a_y\theta\frac{d\theta}{dx} - 2\theta\frac{dv}{dx}$$

and

$$\alpha_x = \frac{1}{I_\omega}\int_{\mathcal{A}}(y^2+z^2)\omega(y,z)d\mathcal{A}, \quad a_y = \frac{1}{J_y}\int_{\mathcal{A}} z(z^2+y^2)d\mathcal{A} - 2z_s, \quad a_z = \frac{1}{J_z}\int_{\mathcal{A}} y(z^2+y^2)d\mathcal{A} - 2y_s$$

$$J_0 = J_y + J_z, \quad r_0 = \sqrt{\frac{J_0}{A} + (y_s)^2 + (z_s)^2}$$

# The elastic energy stored in the bar $W(\boldsymbol{u})$

The elastic energy $W(\boldsymbol{u})$ is assumed in the form:

$$W(\boldsymbol{u}) = \frac{1}{2}\int_0^l \left[ EJ_y \left(\frac{d^2 w}{dx^2}\right)^2 + EJ_z \left(\frac{d^2 v}{dx^2}\right)^2 + GJ \left(\frac{d\theta}{dx}\right)^2 + EI_\omega \left(\frac{d^2 \theta}{dx^2}\right)^2 \right] dx$$

and is treated as functional $W(\boldsymbol{u}) = W(w, v, \theta)$.

# The Rayleigh quotient

Thus, $\lambda_1$ is given by the formula:

$$\frac{1}{\lambda_1} = \max_{w,v,\theta \text{ kin. adm.}} \pm \frac{W_0(w,v,\theta)}{W(w,v,\theta)}$$

called the Rayleigh quotient.

# Axial Buckling

Both ends of the bar are subjected to compression forces $S$. Thus, the initial stress components equal

$$\sigma_0^{11} = \frac{N^0(x)}{A}, \quad \sigma_0^{ij} = 0, \quad (i,j) \neq (1,1)$$

where $N^0(x) = -S = \text{const}$.

$$N^0(x)b_N(x) + M_y^0(x)b_y^M(x) + M_z^0(x)b_z^M(x) + B_x^0(x)b_\omega(x) + \mathcal{M}^0(x)b_{\mathcal{M}}(x) + T_z^0(x)b_z^T(x) + T_y^0(x)b_y^T(x)$$

$$W_0 = \frac{1}{2}\int_0^l \left[ \left(\frac{dw}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + 2z_s \frac{dv}{dx}\frac{d\theta}{dx} - 2y_s \frac{dw}{dx}\frac{d\theta}{dx} + (r_0)^2 \left(\frac{d\theta}{dx}\right)^2 \right] dx$$

# Axial Buckling - bar fork-supported at the ends

$$\det \begin{bmatrix} S_y^{cr} - S & 0 & y_s S \\ 0 & S_z^{cr} - S & -z_s S \\ y_s S & -z_s S & (r_0)^2 (S_s^{cr} - S) \end{bmatrix} = 0 \tag{3}$$

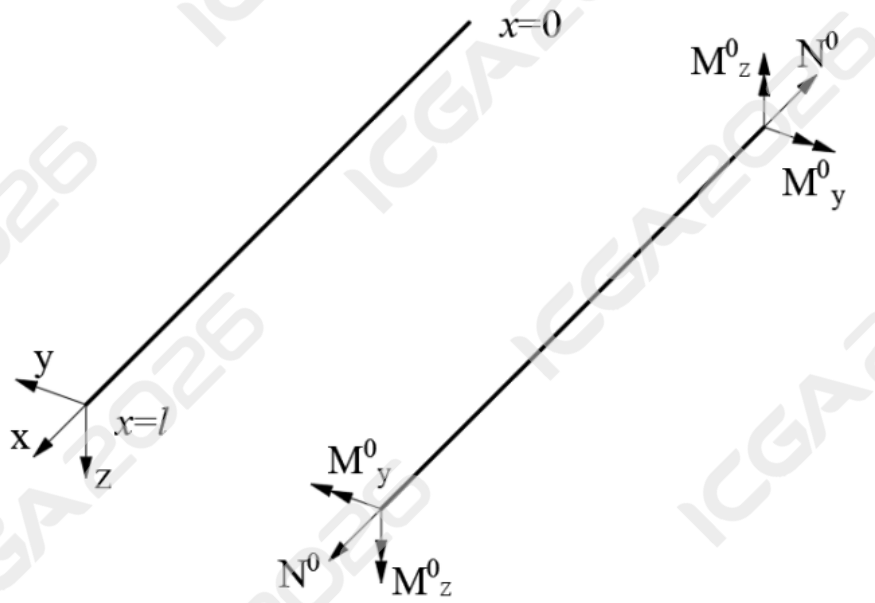
where

$$S_y^{cr} = \frac{\pi^2 E J_y}{l^2}, \quad S_z^{cr} = \frac{\pi^2 E J_z}{l^2}, \quad S_s^{cr} = \frac{1}{(r_0)^2} \left( \frac{\pi^2 E I_\omega}{l^2} + GJ \right)$$

see Trahair et al (2008) in section 3.7.5 Eq.(3.57).

# Buckling of a bar under an axial load and pure bending

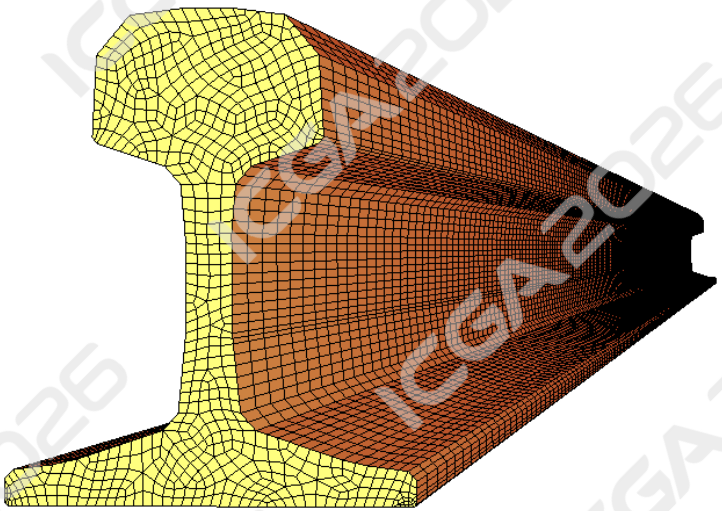
Let $t_y = t_z = 0$ and the component $t_x$ of the traction on the end faces will be of the form $t_x = \pm(a + by + cz)$ on the faces $x = 0$, $x = l$, respectively. The following internal forces appear:
$N^0 = \text{const}, M_y^0 = \text{const}, M_z^0 = \text{const}, T_z^0 = 0, T_y^0 = 0, \mathcal{M}^0 = 0, B^0 = 0.$



# Buckling of a bar under a pure bending

Consider the special case of $t_x = \pm cz$ for $x = 0$, $x = l$ respectively, which gives $N^0 = 0$, $M_z^0 = 0$, $M_y^0 = M^0 \neq 0, B^0 = 0$. The cross section $\mathcal{A}$ is arbitrary.

$$\frac{1}{|\lambda_1|} = \max_{v,\theta \text{ kin. admiss.}} \frac{\int_0^l \left[ a_y \left( \frac{d\theta}{dx} \right)^2 - 2 \frac{dv}{dx} \frac{d\theta}{dx} \right] dx}{\int_0^l \left[ EJ_z \left( \frac{d^2 v}{dx^2} \right)^2 + GJ \left( \frac{d\theta}{dx} \right)^2 + EI_\omega \left( \frac{d^2 \theta}{dx^2} \right)^2 \right] dx}$$



- 49E1 rail. Length 4m. Fork support
- Sofistik: $M_{y_{crit}}^{Sofistik} = 231.9kN$
- Mathematica: $M_{y_{crit}} = 232.9kN$
- Difference 0.43[%]

# Lateral Buckling

Only in case if the bar is subjected to two opposite bending moments at the end the $M_y^0$ and $M_z^0$ are constant and the shear forces $T_y^0$, $T_z^0$ vanish. Only then the stability problem is elementary solvable. Otherwise:

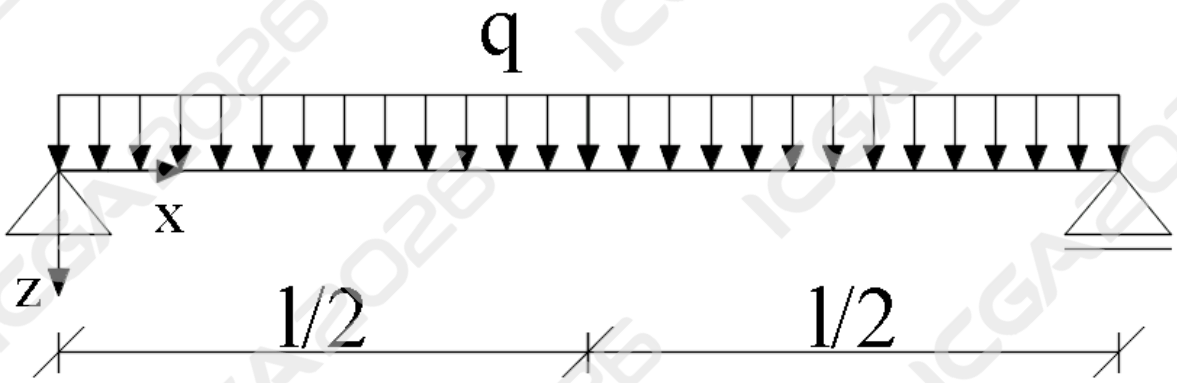
$$W_0 = \frac{1}{2}\int_0^l \left[ M_y^0(x) b_y^M(x) + M_z^0(x) b_z^M(x) + T_y^0(x) b_y^T(x) + T_z^0(x) b_z^T(x) \right] dx + \widetilde{W}_y + \widetilde{W}_z$$

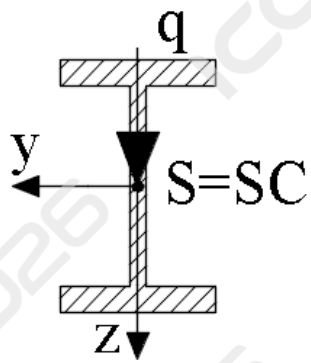
which we can rewrite in the form dependent only on $M_y^0$, $M_z^0$:

$$W_0 = \frac{1}{2}\int_0^l \left[ \frac{d}{dx}(a_y\theta - 2v)\frac{d}{dx}(\theta M_y^0) - \frac{d}{dx}(a_z\theta + 2w)\frac{d}{dx}(\theta M_z^0) \right] dx + \widetilde{W}_y + \widetilde{W}_z$$

# Lateral Buckling, $q = \text{const}$

a) Let us consider a bar of bisymmetric cross-section with both ends fork-supported loaded with $q = \text{const}$ linking points $(x, 0, 0)$ from $x = 0$, $x = l$.




The bending moment is given by $M_y^0(x) = \frac{1}{2} q x (l - x)$.

# Lateral Buckling, $q = \text{const}$

Work of initial forces is given by:

$$W_0 = -\int_0^l \frac{dv}{dx}\frac{d}{dx}\left[\left(\frac{1}{2}qx(l-x)\theta(x)\right)\right]dx = -\frac{1}{2}q\int_0^l \frac{dv}{dx}\frac{d}{dx}\left[x(l-x)\theta(x)\right]dx$$

Functions $v$, $\theta$ are kinematically admissible ($v(0) = 0$, $\theta(0) = 0$; $v(l) = 0$, $\theta(l) = 0$) The Rayleigh quotient takes the form:

$$\frac{1}{|q_{crit}/q_0|} = \max_{\psi,\phi \text{ kin. adm.}} \frac{-\int_0^1 \frac{d\psi}{d\xi}\frac{d}{d\xi}\left[\xi(1-\xi)\phi(\xi)\right]d\xi}{\int_0^1 \left[\left(\frac{d^2\psi}{d\xi^2}\right)^2 + \gamma\left(\frac{d\phi}{d\xi}\right)^2 + \left(\frac{d^2\phi}{d\xi^2}\right)^2\right]d\xi}$$

where $q_0 = \frac{1}{l^4}\sqrt{EJ_zEI_\omega}$, $\xi = \frac{x}{l}$, $\gamma = \frac{l^2GJ}{EI_\omega}$, $\theta = l\sqrt{\frac{EJ_z}{EI_\omega}}\phi$, $\psi = \frac{v}{l}$

# Lateral Buckling, $q = \text{const.}$ Numerical Computation of the Rayleigh quotient

The critical load $q_{crit}$ for the lateral bucking can be found numerically by Mathematica software. Functions $\phi(\xi)$, $\psi(\xi)$ are expanded by the truncated trigonometric series:

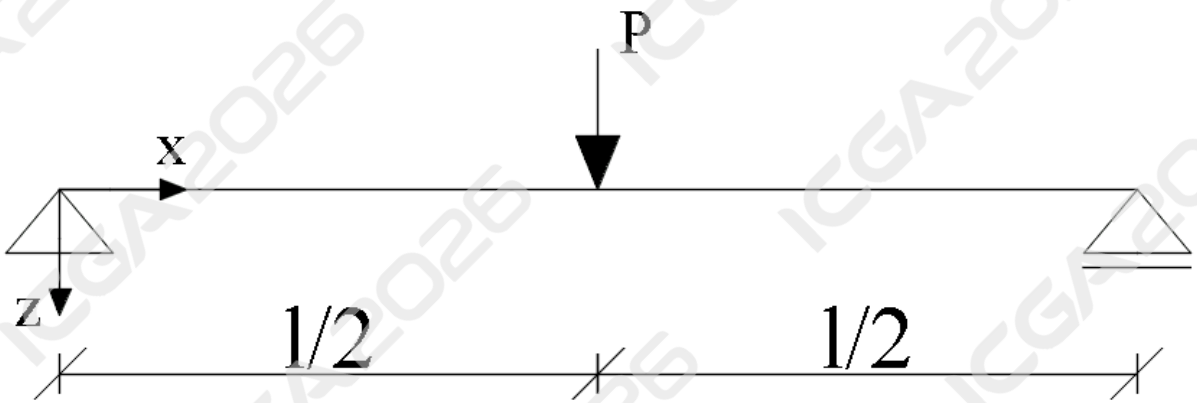
$$\psi(\xi) = \sum_{i=1,3,5}^{n} A_i \sin(i\pi\xi), \quad \phi(\xi) = \sum_{i=1,3,5}^{n} B_i \sin(i\pi\xi), \quad A_i, B_i - \text{unknowns}$$

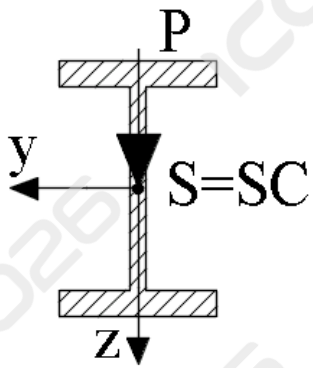
| $\gamma$ | Timoshenko & Gere | Truncated series n=11 | Difference [%] |
|---|---|---|---|
| 0.4 | 143 | 144 | 0.70 |
| 16 | 36.3 | 36.1 | 0.55 |
| 48 | 31.5 | 31.2 | 0.95 |
| 128 | 29.4 | 29.4 | 0.00 |
| 360 | 28.6 | 28.7 | 0.35 |

Values of the ciritical load $q_{crit}/(\sqrt{\gamma}q_0)$.

# Lateral Buckling.
# Pointload $P$ acting in the middle of the bar.

b) Let us consider a bar of bisymmetric cross-section with both ends fork-supported loaded with $P$ acting in the middle of the bar at point $(l/2, 0, 0)$.




$$M_y^0(x) = \begin{cases} \frac{1}{2}Px & 0 < x < \frac{l}{2} \\ \frac{1}{2}P\,(l-x) & \frac{l}{2} < x < l \end{cases}$$

# Lateral Buckling.
# Pointload $P$ acting in the middle of the bar.

$$\frac{1}{|P_{crit}/P_0|} = \max_{\psi,\phi \text{ kin. adm.}} \frac{-\int_0^{1/2} \frac{d\psi}{d\xi}\frac{d}{d\xi}(\xi\phi)d\xi - \int_{1/2}^{1} \frac{d\psi}{d\xi}\frac{d}{d\xi}((1-\xi)\phi)d\xi}{\int_0^1 \left[\left(\frac{d^2\psi}{d\xi^2}\right)^2 + \gamma\left(\frac{d\phi}{d\xi}\right)^2 + \left(\frac{d^2\phi}{d\xi^2}\right)^2\right] d\xi}$$

$$P_0 = \frac{1}{l^3}\sqrt{EJ_z \cdot EI_\omega},\ \xi = \frac{x}{l},\ \gamma = \frac{l^2 GJ}{EI_\omega}\ \theta = l\sqrt{\frac{EJ_z}{EI_\omega}}\phi,\ \psi = \frac{v}{l}$$

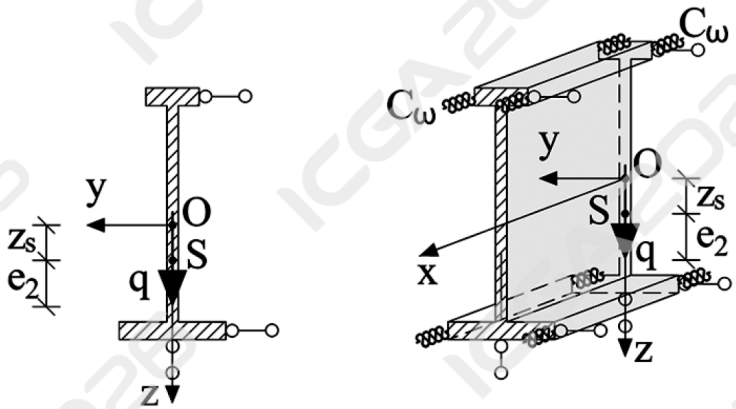
# Lateral Buckling.
# Pointload $P$ acting in the middle of the bar.
# Numerical Computation of the Rayleigh quotient.

| $\gamma$ | Timoshenko & Gere | Truncated series n=11 | Difference [%] |
|---|---|---|---|
| 0.4 | 86.4 | 86.9 | 0.58 |
| 16 | 21.8 | 21.8 | 0.00 |
| 48 | 18.8 | 18.7 | 0.53 |
| 96 | 17.9 | 17.9 | 0.00 |
| 320 | 17.2 | 17.2 | 0.00 |

Values of the ciritical load $P_{crit}/(\sqrt{\gamma}P_0)$.

# Lateral Buckling

Czubacki, R., Lewiński, T. Application of the Rayleigh quotient method in the analysis of stability of straight elastic bars. Archives of Civil Engineering 70(4), 85–98 (2024). https://doi.org/10.24425/ace.2024.151881




$$C_\omega = \frac{1}{3} G b \left(t_p\right)^3 h_s$$

$$K_\omega = \frac{l C_\omega}{2 E I_\omega + l C_\omega}$$

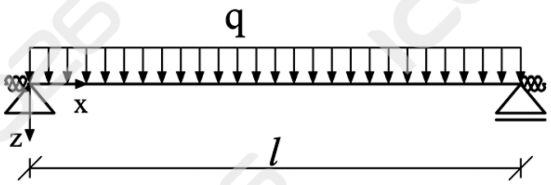
Fig. 2. Parametrization of the bar's mono-symmetric cross section; $y$ and $z$ are principal axes; position of the centroid (O) and the shear center (S). Possible point of application of the transverse load $q_z = q$ has coordinates: $(x, 0, z_2)$ $z_2 = e_2 + z_S$

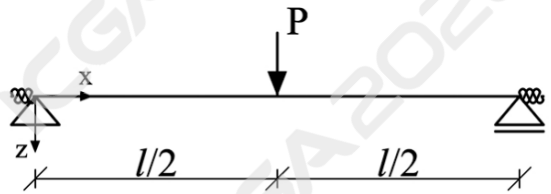
$$\frac{1}{|q_{\text{crit}}|} = \max_{\substack{\text{over } \theta, v \text{ being} \\ \text{kinematically} \\ \text{admissible}}} \frac{\int_0^l \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}a_y\theta - v\right)\frac{\mathrm{d}}{\mathrm{d}x}\left[x\left(l-x\right)\theta\right]\mathrm{d}x + \int_0^l e_2\theta^2\mathrm{d}x - \int_0^l \left(\frac{1}{2}a_y\right)\theta^2\mathrm{d}x}{\int_0^l \left[EJ_z\left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\right)^2 + GJ\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + EI_\omega\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\right)^2\right]\mathrm{d}x + C_\omega\left(\left(\left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{x=0}\right)^2 + \left(\left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{x=l}\right)^2\right)}$$

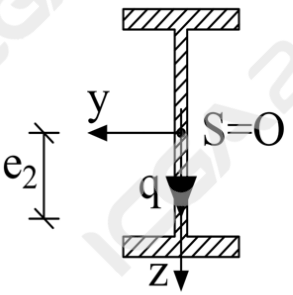
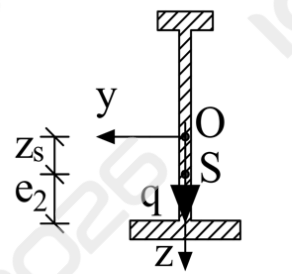
$$\frac{1}{|P_{\text{crit}}|} = \max_{\substack{\text{over } \theta, v \text{ being} \\ \text{kinematically} \\ \text{admissible}}} \frac{\int_0^{\frac{l}{2}} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}a_y\theta - v\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x\theta\right)\mathrm{d}x + \int_{\frac{l}{2}}^l \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}a_y\theta - v\right)\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(l-x\right)\theta\right]\mathrm{d}x + \left(\left(e_2 - \frac{1}{2}a_y\right)\theta^2\right)\Big|_{x=\frac{l}{2}}}{\int_0^l \left[EJ_z\left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\right)^2 + GJ\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + EI_\omega\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\right)^2\right]\mathrm{d}x + C_\omega\left(\left(\left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{x=0}\right)^2 + \left(\left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{x=l}\right)^2\right)}$$

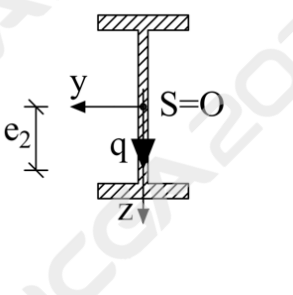
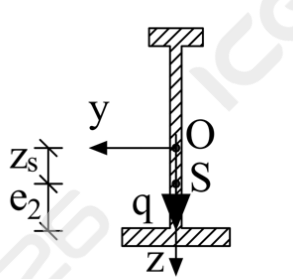
# Lateral Buckling

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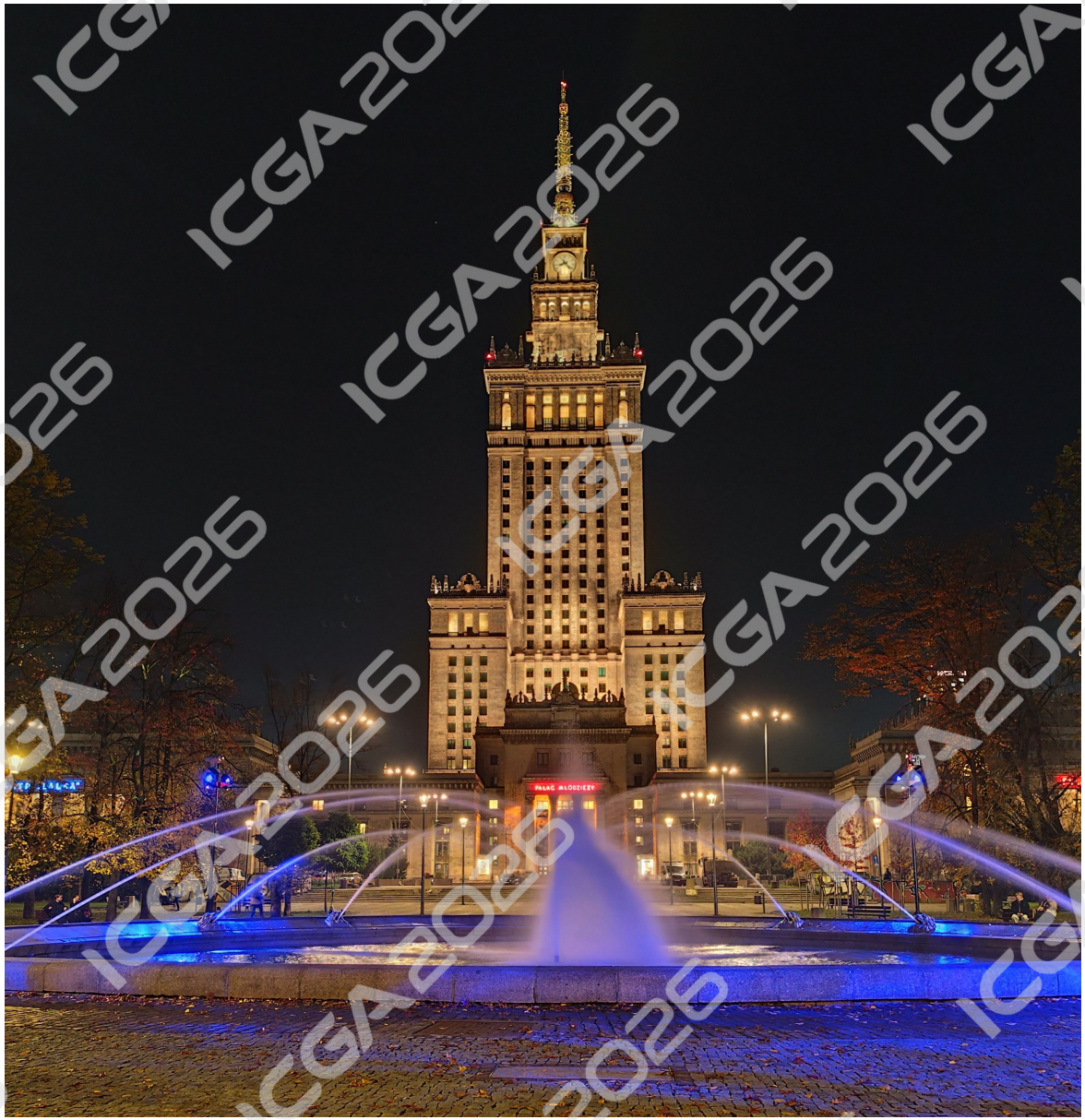

| Cross-section | $K_\omega$ | $e_2$ [m] | $M_{cr}$ LTBeamN | $M_{cr}$ Raylaigh quotient | % |
|---|---|---|---|---|---|
|  | 0 | -0.25 | 239.01 | 238.97 | 0.02 |
| | | 0 | 316.26 | 316.24 | 0.01 |
| | | 0.25 | 418.25 | 418.17 | 0.02 |
| | 0.25 | -0.25 | 252.21 | 252.36 | -0.06 |
| | | 0 | 329.72 | 329.80 | -0.02 |
| | | 0.25 | 430.56 | 430.67 | -0.03 |
| | 0.5 | -0.25 | 273.48 | 273.78 | -0.11 |
| | | 0 | 351.15 | 351.36 | -0.06 |
| | | 0.25 | 450.37 | 450.61 | -0.05 |
| | 0.75 | -0.25 | 313.46 | 314.81 | -0.43 |
| | | 0 | 391.05 | 392.35 | -0.33 |
| | | 0.25 | 487.57 | 488.73 | -0.24 |
| | 1 | -0.25 | 418.76 | 422.78 | -0.96 |
| | | 0 | 495.20 | 499.06 | -0.78 |
| | | 0.25 | 585.28 | 588.98 | -0.63 |
|  | 0 | -0.2621 | 113.57 | 113.57 | 0.00 |
| | | 0 | 156.20 | 156.20 | 0.00 |
| | | 0.0379 | 163.81 | 163.81 | 0.00 |
| | 0.25 | -0.2621 | 114.51 | 114.53 | -0.02 |
| | | 0 | 157.05 | 157.06 | -0.01 |
| | | 0.0379 | 164.62 | 164.64 | -0.01 |
| | 0.5 | -0.2621 | 116.17 | 116.19 | -0.02 |
| | | 0 | 158.52 | 158.55 | -0.02 |
| | | 0.0379 | 166.05 | 166.07 | -0.01 |
| | 0.75 | -0.2621 | 119.82 | 119.95 | -0.11 |
| | | 0 | 161.77 | 161.89 | -0.07 |
| | | 0.0379 | 169.17 | 169.29 | -0.07 |
| | 1 | -0.2621 | 134.48 | 135.26 | -0.58 |
| | | 0 | 174.50 | 175.16 | -0.38 |
| | | 0.0379 | 181.42 | 182.04 | -0.34 |

| Cross-section | $K_\omega$ | $e_2$ [m] | $M_{cr}$ LTBeamN | $M_{cr}$ Raylaigh quotient | % |
|---|---|---|---|---|---|
|  | 0 | -0.25 | 269.61 | 269.61 | 0.00 |
| | | 0 | 380.89 | 380.69 | 0.05 |
| | | 0.25 | 533.83 | 534.36 | -0.10 |
| | 0.25 | -0.25 | 283.85 | 283.89 | -0.01 |
| | | 0 | 396.10 | 396.50 | -0.10 |
| | | 0.25 | 549.91 | 550.55 | -0.12 |
| | 0.5 | -0.25 | 306.31 | 306.54 | -0.08 |
| | | 0 | 421.19 | 421.51 | -0.08 |
| | | 0.25 | 576.21 | 576.22 | 0.00 |
| | 0.75 | -0.25 | 348.01 | 349.31 | -0.37 |
| | | 0 | 467.05 | 468.68 | -0.35 |
| | | 0.25 | 624.11 | 625.35 | -0.20 |
| | 1 | -0.25 | 453.80 | 458.20 | -0.97 |
| | | 0 | 584.79 | 588.86 | -0.70 |
| | | 0.25 | 748.93 | 753.16 | -0.56 |
|  | 0 | -0.2621 | 125.70 | 125.76 | -0.05 |
| | | 0 | 193.00 | 193.08 | -0.04 |
| | | 0.0379 | 206.08 | 206.14 | -0.03 |
| | 0.25 | -0.2621 | 126.56 | 126.59 | -0.02 |
| | | 0 | 194.04 | 194.05 | -0.01 |
| | | 0.0379 | 207.06 | 207.14 | -0.04 |
| | 0.5 | -0.2621 | 127.98 | 128.01 | -0.02 |
| | | 0 | 195.75 | 195.74 | 0.01 |
| | | 0.0379 | 208.77 | 208.84 | -0.03 |
| | 0.75 | -0.2621 | 131.18 | 131.30 | -0.09 |
| | | 0 | 199.41 | 199.55 | -0.07 |
| | | 0.0379 | 212.55 | 212.71 | -0.08 |
| | 1 | -0.2621 | 143.99 | 144.71 | -0.50 |
| | | 0 | 213.98 | 214.75 | -0.36 |
| | | 0.0379 | 227.37 | 228.09 | -0.32 |

# Conclusions

The Rayleigh quotient delivers the critical load as well as the form of the buckling directly. This equation is the most compact formulation of the buckling problem of a straight bar and is a convienient starting point for the numerical computations. Let us recall that only if all the forces $N^0(x), \ldots, T_y^0(x)$ are constant the buckling problem can be analytically solved. Thus, almost nothing is lost if the Rayleigh quotient is used instead of a cumbersome analysis of the set of the relevant equilibrium equations.

# *Thank you for your kind attention!*




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